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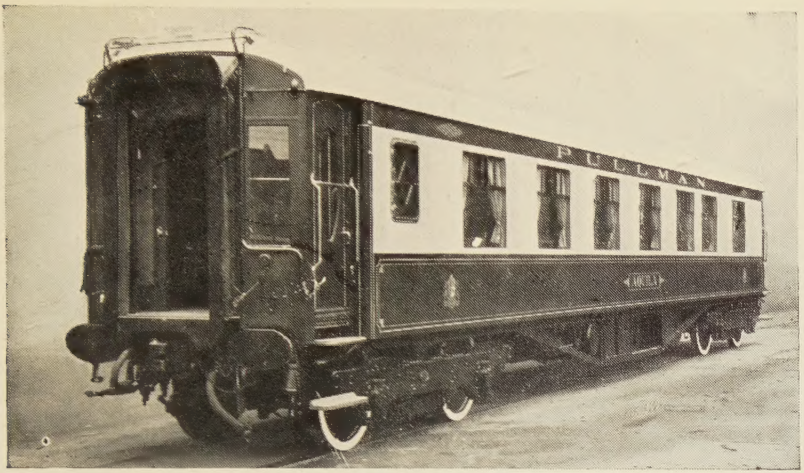

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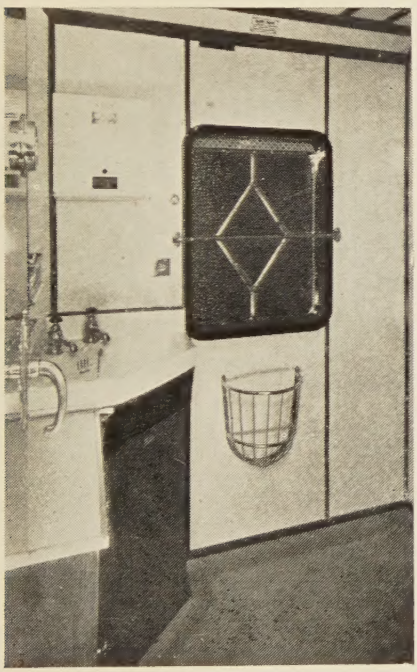

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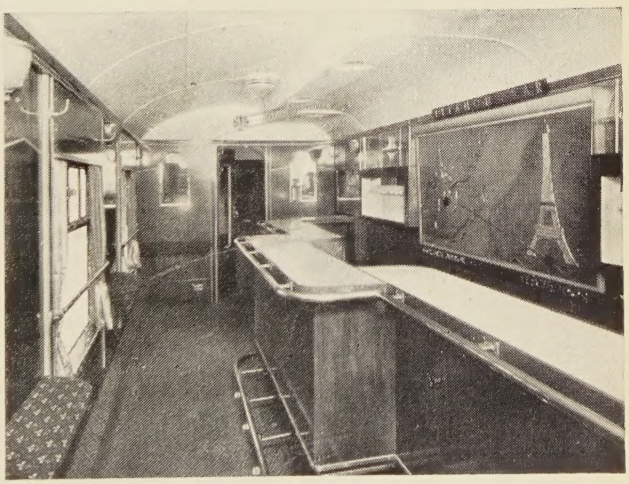

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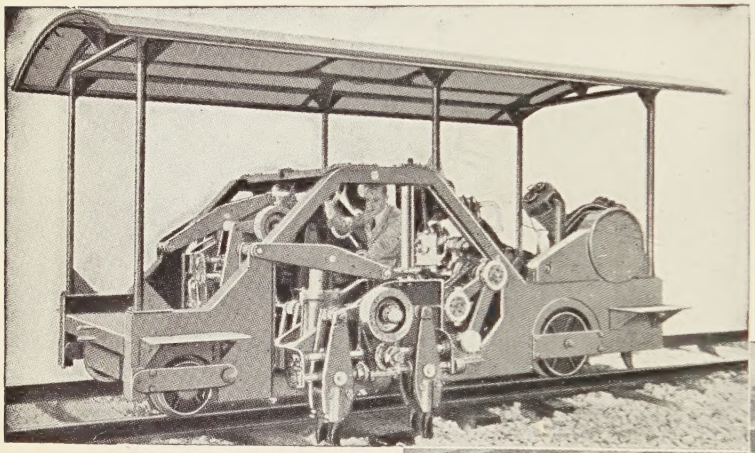

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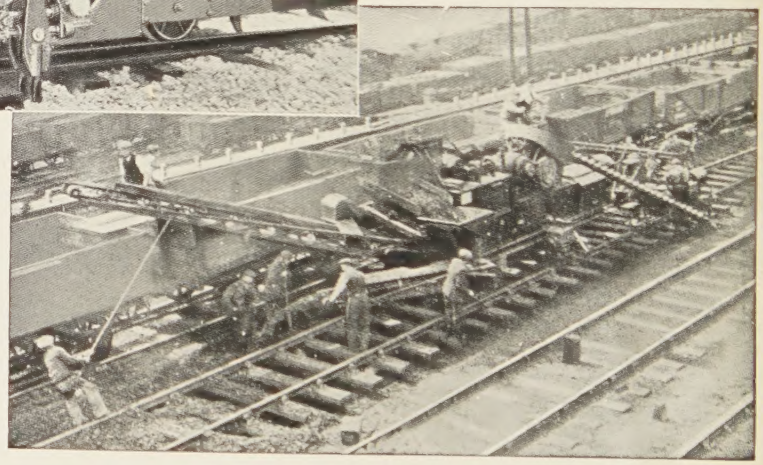

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MONTHLY BULLETIN

OF THE

INTERNATIONAL RAILWAY CONGRESS ASSOCIATION

[385. (05]

VOL. XXIX. — 1952.

NOTICE TO READERS

LIST OF ABBREVIATIONS OF MEASURES, ETC., USED IN THE *BULLETIN*

Kilometre	km	Ampere	A
Metre	m	Kiloampere	kA
Centimetre	cm	Milliampere	mA
Square kilometre	km^2	Kilowatt	kW
Square metre	m^2	Kilowatt-hour	kWh
Square centimetre	cm^2	Watt	W
Hectare	ha	Megohm	MΩ
Are	a	Ohm	Ω
Centiare	ca	Ampere-hour	Ah
Cubic metre	m^3	Microfarad	μF
Cubic centimetre	cm	Henry	H
Stere	st	Weber	Wb
Hectolitre	hl	Hertz	Hz
Litre	l	Calory	Cal
Centilitre	cl	Kilo-calory	kcal
Degree	°	Lumen	lm
Minute	′	Lux	lx
Second	″	Horse-power	HP
Tonne (Metric)	t	Per cent	%
Ton (English)	ton	Per thousand	‰
Hundredweight	cwt	Tonne-kilometre	tkm
Kilogramme	kg	Passenger-kilometre	p. km
Gramme	g	Train-kilometre	tr. km
Centigramme	cg	Franc	fr.
Hour	h	Lire	L.
Minute	min	Pound sterling	£
Second	s	Shilling	s.
Kilogrammetre	kgm	Penny	d.
Volt	V	Dollar	$
Millivolt	mV	Cent	ct
Kilovolt	kV	Deutsche Mark	D. M.
		Pfennig	pf

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SJÖBERG (Arne). — **Marginal costs — a 40-year old management tool for rate fixing on the Swedish State Railways.** (9 000 words, tables & fig.)

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Bull. of the Int. Ry. Congr. Ass., January, p. 75.
OBITUARY NOTICE : Axel GRANHOLM. (600 words & photograph.)

Vol. XXIX. — No. 1. January 1952.

BULLETIN
OF THE
INTERNATIONAL RAILWAY CONGRESS
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(ENGLISH EDITION)

[625 .14 (01]

Calculation of bars of infinite length on resilient bearings, with notes on the depression given to a bearing by loading another bearing,

by Ir. W. J. VAN DER EB,
Engineer at the « Organisatie voor Toegepast-Natuurwetenschappelijk Onderzoek »,
and Dr. Ir. A. D. DE PATER,
Engineer, Netherlands Railways.

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1. Introduction.

In the *Congress Bulletin* of May, 1948, appeared an article by Dr. Eng. A. D. DE PATER (1) in which a method of calculation was shewn for a prismatic bar of infinite length on equidistant bearings. Such a system may apply particularly as a model replacing a pair of rails on resilient sleepers.

It later appeared that the question had already been dealt with, apart from the article already quoted, by DESPRETS (1), by CZITARY (2) and also by RITTER (3).

In the calculations now undertaken, we

(1) A.D. DE PATER : « Calculations for bars of infinite length with elastic supports », *I.R.C. Bulletin*, vol. 25 (1948), p. 265. This article will be referred to as Article 1.

(1) R. DESPRETS : « Notes on rail calculations », *I.R.C. Bulletin*, No. 5 (1921), p. 507.

(2) E. CZITARY : « Beitrag zur Berechnung des Querschwellenoberbaues », *Organ für die Fortschritte des Eisenbahnwesens*, vol. 91 (1936), p. 154.

(3) W. RITTER : « Anwendungen der graphischen Statik », Zürich, vol. III (1900), p. 167.

shall once more shew a method of calculation for the system we have described; this time we shall use the calculation of variations. We shall see that this also leads to a fairly simple solution, but this fact alone would not justify the presentation of a further article dealing with the same problem. However, it would seem that the use of calculation of variations provides for an important phenomenon which occurs in a superstructure but which has so far been completely ignored in similar calculations. The phenomenon mentioned is that when, in a series of sleepers detached from the rails, a single one is loaded, not only this one but also the neighbouring ones are depressed, as we shall shew in more detail in Chapter **2.**

As in Article 1, we shall confine ourselves to the case in which the bar is loaded with a single force. When the bar is subject to more forces (in the case of continuous loading) owing to the fact that the bar is of infinite length, the solution can be found by superimposing the appropriate solutions for each force (in the case of partial force of a continuous load).

2. Relation between the forces operating on a bearing and the amount of depression.

DRIESSEN (1) has recorded his experiences, in which a sleeper, detached from the two rails, was loaded with two equal forces, bearing on the sleeper in the position occupied by the rails. He then measured the depression of the loaded sleeper and also the depression of the neighbouring sleepers, which were also detached from the rail.

The depressions recorded had approximately the dimensions shewn in the sketch in figure 1. In this sketch the sleepers are indicated by numbers and the sleeper marked m is the loaded one. P_m is the sum of the two forces on this sleeper. We will now attempt to present such a series of depressions by a mathematical formula.

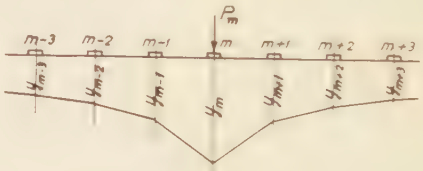


Fig. 1. — Depression of a loaded sleeper and neighbouring sleepers.

As in Article 1, we shall not deal with the extent to which Hooke's law applies to the construction of a bearing created by the sleeper and sub-soil, and assume tacitly that the depression of each sleeper is directly proportionate to the load P_m. It is thus logical to represent the depression y_n of the sleeper n, introducing two constants $\bar{\mu}$ and λ, by :

$$\left.\begin{aligned} y_n &= \mu\ \lambda^{m-n}\ P_m \text{ for } m \geqslant n, \\ y_n &= \bar{\mu}\ \lambda^{n-m}\ P_m \text{ for } m \leqslant n; \end{aligned}\right\} \quad . \quad . \quad (1)$$

In these formulae λ is a number without dimension < 1 and $\bar{\mu}$ a constant with the dimension ($\bar{\mu}$ = length : force). In effect, on one hand the formulae (1) represent fairly accurately the depressions of the sleepers and on the other hand a relatively simple formula can be found which gives

(1) Ch. H. J. DRIESSEN : « Nieuwe inzichten bij de berekening van den bovenbouw der spoorwegen », *De Ingenieur*, vol. 51 (1936), p. V 75, fig. 21.

the relationship between each depression and load, when each sleeper is loaded.

Since there is, more usually, in the latter case; instead of (1) :

$$y_n = \bar{\mu}\left(\sum_{m=-\infty}^{n-1} \lambda^{n-m} P_m + \sum_{m=n}^{\infty} \lambda^{m-n} P_m\right) \quad \ldots\ldots\ldots (2)$$

We may calculate $\delta^2 y_n = y_{n+1} - 2\,y_n + y_{n-1}$. We find successively :

$$y_{n+1} - y_n = \bar{\mu}(1-\lambda)\left(-\sum_{m=-\infty}^{n} \lambda^{n-m} P_m + \sum_{m=n+1}^{\infty} \lambda^{m-n-1} P_m\right),$$

$$y_n - y_{n-1} = \bar{\mu}(1-\lambda)\left(-\sum_{m=-\infty}^{n-1} \lambda^{n-m-1} P_m + \sum_{m=n}^{\infty} \lambda^{m-n} P_m\right),$$

$$\delta^2 y_n = \frac{\bar{\mu}(1-\lambda)^2}{\lambda}\left(\sum_{m=-\infty}^{n-1} \lambda^{n-m} P_m + \sum_{m=n+1}^{\infty} \lambda^{m-n} P_m\right) - 2\,\bar{\mu}(1-\lambda) P_n,$$

or

$$\delta^2 y_n = \frac{\bar{\mu}(1-\lambda)^2}{\lambda}\left(\sum_{m=-\infty}^{n-1} \lambda^{n-m} P_m + \sum_{m=n}^{\infty} \lambda^{m-n} P_m\right) - \frac{\bar{\mu}(1-\lambda^2)}{\lambda} P_n. \quad \ldots\ldots (3)$$

The elimination of the expression in brackets with the signs Σ of (2) and (3) gives, for the desired formula :

$$-\lambda\,\delta^2 y_n + (1-\lambda)^2 y_n = \bar{\mu}(1-\lambda^2) P_n. \quad \ldots\ldots\ldots (4)$$

Formula (1) and the formulae derived from it are also valid for the special case, dealt with in Article 1, where a sleeper loaded singly is subjected to depression. Proceeding to this case, it is sufficient to allow $\lambda \to 0$ in the formulae; (1) and (4) then both become :

$$y_n = \bar{\mu} P_n. \quad \ldots\ldots\ldots\ldots\ldots (5)$$

3. Derivation of the equation for specific degrees of bending of a bar.

Let us now assume that on the resilient bearings mentioned in Chapter 2 is located a pair of wheels of infinite length with a modulus of elasticity E and a constant moment of inertia I. For ease of expression, we shall again refer in the following to the assembly of two rails as the « bar » and also to « resilient bearings ». In addition,

we shall suppose that the bar is loaded with a single force P in the span between the bearings 0 and 1; we shall call x_p the distance from the force to the bearing 0 (see fig. 2) and state :

$$x = l\xi, \quad x_p = l\xi_p, \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

formulae in which x_p represents the coordinate which indicates the point in the bar at which the load is measured.

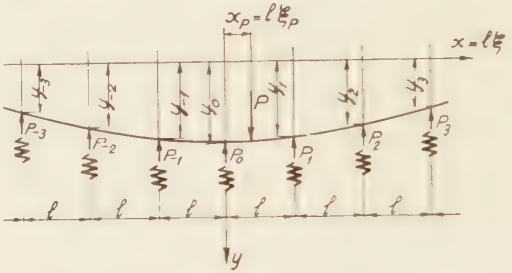


Fig. 2. — Bar of infinite length resting on resilient bearings, loaded with a force P.

We will let M_n be the bending moments of the bar in the region of the bearing n. This will be positive when the bar is loaded from above at the position of the moment. From the equilibrium of the parts of the bar in the spans $(n-1)-n$ and $n-(n+1)$ can readily be deduced the relationship between M_n and P_n.

$$\delta^2 M_n = -P_n l \text{ for } n \neq 0, 1 \ldots\ldots\ldots\ldots (2)$$

In these formulae n must not equal 0 or 1; for bearings having these values of n, we can take the equations :

$$\delta^2 M_0 = P_0 l + P l(1-\xi_p), \quad \delta^2 M_1 = -P_1 l + P l \xi_p \ldots\ldots (3)$$

Beside the equation (2, 4) which indicates the relation between the deformation and the load on the bearings, and the formula (2), which is derived from the equilibrium of the bar, we can write another formula concerning the deformation of the bar. In fact, we find with the equation, the three moments (1) :

$$\frac{\nu}{l}(M_{n-1} + 4M_n + M_{n+1}) = \delta^2 y_n \text{ for } n \neq 0, 1, \ldots\ldots\ldots (4)$$

$$\left.\begin{aligned} \frac{\nu}{l}(M_{-1} + 4M_0 + M_1) &= \delta^2 y_0 + \nu P \xi_p (1-\xi_p)(2-\xi_p), \\ \frac{\nu}{l}(M_0 + 4M_1 + M_2) &= \delta^2 y_1 + \nu P \xi_p (1-\xi_p^2). \end{aligned}\right\} \ldots\ldots (5)$$

(1) See for example : C.B. BIEZENO and R. GRAMMEL : *Technische Dynamik*, Berlin (1939), p. 229.

In these formulae, we have, for purposes of brevity, taken as in Article 1

$$\nu = \frac{l^3}{6\,EI}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

they can be further somewhat reduced under the form :

$$\frac{\nu}{l}\delta^2 M_n + 6\frac{\nu}{l}M_n = \delta^2 y_n \text{ for } n \neq 0,\ 1, \quad \ldots\ldots\ldots\ldots (7)$$

$$\left.\begin{aligned} &\frac{\nu}{l}\delta^2 M_o + 6\frac{\nu}{l}M_o = \delta^2 y_o + \nu P\xi_p(1-\xi_p)(2-\xi_p), \\ &\frac{\nu}{l}\delta^2 M_1 + 6\frac{\nu}{l}M_1 = \delta^2 y_1 + \nu P\xi_p(1-\xi_p{}^2). \end{aligned}\right\} \quad \ldots\ldots (8)$$

So far, we have eliminated P_n from equation (**2**, 4) and (2) and where applicable from (3), in order to derive from it the relationship of (7) and (8) between M_n and y_n. For this, we find, if we postulate as in Article 1 :

$$\bar{\mu} = \frac{\nu}{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

(C being a factor without dimension),

$$\frac{1-\lambda^2}{C}\,\frac{\nu\delta^2 M_n}{l} = \lambda\delta^2 y_n - (1-\lambda)^2 y_n \text{ for } n \neq 0,\ 1, \ldots\ldots (10)$$

$$\left.\begin{aligned} &\frac{1-\lambda^2}{C}\,\frac{\nu\delta^2 M_o}{l} - \frac{1-\lambda^2}{C}\,\nu P(1-\xi_p) = \lambda\delta^2 y_o - (1-\lambda)^2 y_o, \\ &\frac{1-\lambda^2}{C}\,\frac{\nu\delta^2 M_1}{l} - \frac{1-\lambda^2}{C}\,\nu P\xi_p = \lambda\delta^2 y_1 - (1-\lambda)^2 y_1. \end{aligned}\right\} \quad (11)$$

In the following Chapter we shall consider the general solutions of equations (7) and (10).

4. General solution of the equation for specific variations.

We now substitute as a solution of the equations (**3**, 7) and (**3**, 10) :

$$y_n = \varepsilon^n, \quad \frac{\nu M_n}{l} \quad \gamma\varepsilon'', \quad \ldots\ldots\ldots\ldots\ldots (1)$$

ε and γ being numbers without dimension. Introducing the quantity

$$\theta = \frac{(1-\varepsilon)^2}{\varepsilon} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

we have

$$\delta^2 y_n = \theta \varepsilon^n, \quad \frac{\nu \delta^2 M_n}{l} = \gamma \theta \varepsilon^n. \quad \ldots\ldots\ldots\ldots\ldots (3)$$

By substituting this in (3, 7) and (3, 10), we find for θ and γ the two equations

$$\gamma(6+\theta) = \theta, \quad \frac{1-\lambda^2}{C}\gamma\theta = \lambda\theta - (1-\lambda)^2, \quad \ldots\ldots\ldots (4)$$

from which an equation can be derived, comprising θ only :

$$\left(\frac{1-\lambda^2}{C} - \lambda\right)\theta^2 - (1 - 8\lambda - \lambda^2)\theta + 6(1-\lambda)^2 = 0. \quad \ldots\ldots (5)$$

The solution of this is :

$$\theta_{1,2} = \frac{-(1-8\lambda+\lambda^2) \mp \sqrt{(1-8\lambda+\lambda^2)^2 - 24(1-\lambda)^2\left(\frac{1-\lambda^2}{C} - \lambda\right)}}{2\left(\frac{1-\lambda^2}{C} - \lambda\right)}; \quad \ldots (6)$$

Consequently from (2) there are four solutions for ε :

$$\left.\begin{aligned} \varepsilon_{1,3} &= 1 + \frac{1}{2}\theta_1 \mp \frac{1}{2}\sqrt{\theta_1(\theta_1+4)}, \\ \varepsilon_{2,4} &= 1 + \frac{1}{2}\theta_2 \mp \frac{1}{2}\sqrt{\theta_2(\theta_2+4)}, \end{aligned}\right\} \quad \ldots\ldots\ldots (7)$$

in which are equal

$$\varepsilon_3 = \varepsilon_1^{-1}, \quad \varepsilon_4 = \varepsilon_2^{-1}. \quad \ldots\ldots\ldots\ldots (8)$$

It is to be expected that for certain values of C and λ, some of the quantities $\theta_{1,2}$ and $\varepsilon_{1\ldots4}$ will be unreal. To investigate this, we trace a diagram (C, λ). Given that on the values of C > 0 and $0 < \lambda < 1$ have a physical meaning, it is necessary to consider only the first quadrant and of this only the region $\lambda < 1$. In this diagram (fig. 3) we trace several curves which, in addition to indicating the outline of the

formulae (6) and (7), will give appropriate values for $\theta_{1,2}$ and/or ε_{1-4}, i.e. the curves $a - d$, defined respectively by the equations :

$$a : \quad (1 - 8\lambda + \lambda^2)^2 - 24\,(1-\lambda)^2\left(\frac{1-\lambda^2}{C} - \lambda\right) = 0, \quad \ldots \quad (9^a)$$

$$b : \quad \frac{1-\lambda^2}{C} - \lambda = 0, \quad \ldots \quad (9^b)$$

$$c : \quad \lambda = 1, \quad \ldots \quad (9^c)$$

$$d : \quad \lambda = \lambda_1, \quad \ldots \quad (9^d)$$

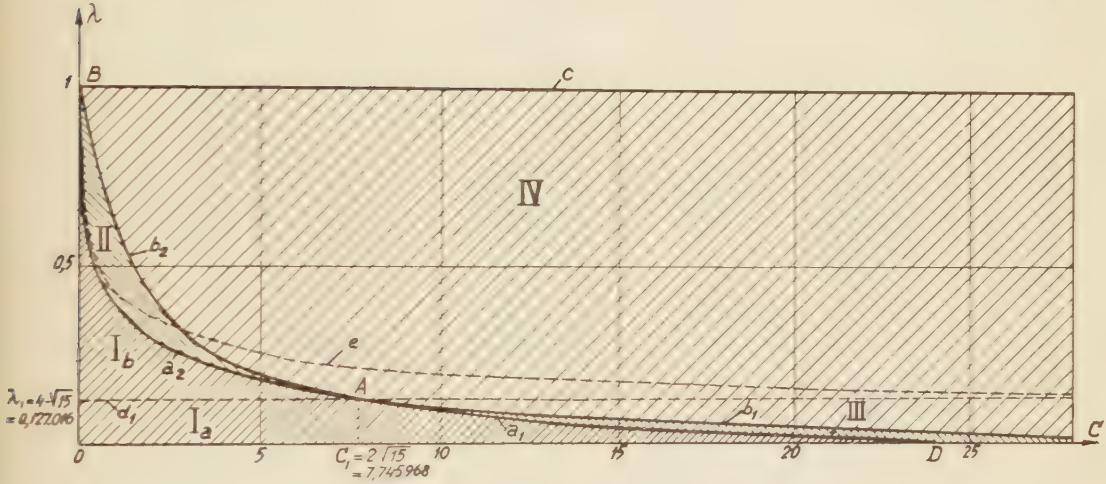


Fig. 3. — Plan (C, λ).

in which λ_1 equals the root located between 0 and 1 of the equation

$$\lambda^2 - 8\lambda + 1 = 0, \quad \ldots \quad (10)$$

so that

$$\lambda_1 = 4 - \sqrt{15} = 0.127016.$$

The broken curve e which is also found in figure 3 will be interpreted in Chapter 8. Curves *a* and *b* can easily be drawn by resolving equations 9^a and 9^b, C being a function of λ. We find, respectively :

$$C = \frac{1-\lambda^2}{\lambda + \frac{1}{24}\left(\frac{1 - 8\lambda + \lambda^2}{1-\lambda}\right)^2} \quad \text{for curve } a, \quad \ldots \quad (11^a)$$

$$C = \frac{1-\lambda^2}{\lambda} \quad \text{for curve } b. \quad \ldots \quad (11^b)$$

Table I gives some values of λ and C. In addition, it is obvious that the curves *a* and *b* are tangential to each other at the point A (C_1, λ_1) and meet at the point B (0, 1). The parts of these curves which refer to the values $\lambda < \lambda_1$ and $\lambda > \lambda_1$ are indicated respectively by the indices 1 and 2.

TABLE I.

Some values of C and λ relating to curves *a* and *b*.

λ	C	
	Curve *a*	Curve *b*
0	24	∞
0.1	8.069	9.9
$4 - \sqrt{15} = 0.127\,016$	$2\sqrt{15} = 7.746$	7.746
0.2	4.355	4.8
0.3	2.041	3.0333
0.4	0.9527	2.1
0.5	0.4260	1.5
0.6	0.1738	1.0667
0.7	0.0599	0.7286
0.8	0.0148	0.45
0.9	0.0016	0.2111
1	0	0

For compound values of $\theta_{1,2}$ we will state :

$$\theta_{1,2} = \frac{-(1 - 8\lambda + \lambda^2) \pm i\sqrt{24\,(1-\lambda)^2\left(\frac{1-\lambda^2}{C} - \lambda\right) - (1 - 8\lambda + \lambda^2)^2}}{2\left(\frac{1-\lambda^2}{C} - \lambda\right)} \qquad (6^a)$$

By calculation, we get from formulae (6) and (6^a) respectively :

$$\theta_1 = \theta_2 = -\frac{12\,(1-\lambda)^2}{1 - 8\lambda + \lambda^2} \quad \text{for curve } a, \qquad (12^a)$$

$$\left.\begin{aligned} &\theta_1 = \pm\infty, \quad \theta_2 = -\frac{6\,(1-\lambda)^2}{1 - 8\lambda + \lambda^2} \text{ for curve } b_1, \\ &\theta_1 = -\frac{6\,(1-\lambda)^2}{1 - 8\lambda + \lambda^2}, \quad \theta_2 = \pm\infty \text{ for curve } b_2 \end{aligned}\right\} \qquad (12^b)$$

(to arrive at this formula, series development is used);

$\theta_1 = \theta_2 = \theta_3 = 6$ for curve c; $\qquad (12^c)$

$$\theta_{1,2} = \pm i \frac{(1-\lambda_1)\sqrt{6}}{\sqrt{\frac{1-\lambda_1^2}{C} - \lambda_1}} \quad \text{for curve } d_1, \ldots \ldots \ldots \ldots$$

$$\theta_{1,2} = \pm \frac{(1-\lambda_1)\sqrt{6}}{\sqrt{\lambda_1 - \frac{1-\lambda_1^2}{C}}} \quad \text{for curve } d_2; \ldots \ldots \ldots \ldots \quad (12^d)$$

$$\theta_1 = \pm \infty, \quad \theta_2 = \pm \infty \quad \text{at point A,} \ldots \ldots \ldots \ldots \quad (12^e)$$

$$\theta_1 = 0, \quad \theta_2 = \frac{6}{2\frac{1-\lambda}{C} - 1} \quad \text{at point B.} \ldots \ldots \ldots \ldots \quad (12^f)$$

The curve defined by $\theta_1 = -4$ or $\theta_2 = -4$ and which may also be relevant, as indicated by formula (7) is equated in the diagram (C, λ) :

$$\left(\frac{1-\lambda^2}{C} - \lambda\right)\left(1+\lambda\right)\left(1+\lambda+8\frac{1-\lambda}{C}\right) = 0, \ldots \ldots \quad (13)$$

which is obtained by calculation from (6), substituting in this equation $\theta_{1,2} = -4$ and eliminating the expression under the root. The partial curve, which is defined by the equation :

$$\frac{1-\lambda^2}{C} - \lambda = 0$$

is the curve b, but from (12^b), it is necessary to use throughout this $\theta_{1,2} \neq -4$ when $0 \leq \lambda_1 \leq 1$: this curve has been introduced by quadrature and must not therefore be taken into consideration. The other partial curves do not fall within the relevant region of the plan (C, λ).

Curves a and b, however, belong to the relevant part of the plan (C, λ) in four regions : I-IV; the following are valid for this, according to (6), (6^a), (12^a) and (12^b).

for region I
- on curve a . . . . . $I_m\theta_{1,2} = 0$;
- on curve d_1 . . . . . $R_e\theta_{1,2} = 0$;
- in the other parts of the region. . :
  - in the subregion I_a : $R_e\theta_{1,2} < 0$;
  - in the subregion I_b : $R_e\theta_{1,2} > 0$.

for region II:

- at point A $\theta_1 = +\infty$, $\theta_2 = +\infty$,
- at point B:
  - $\theta_1 = 0$, $\theta_2 = 0$ for curve a_2;
  - $\theta_1 = 0$, $\theta_2 = +\infty$ for curve b_2;
  - $\theta_1 = 0$, $\theta_2 = \dfrac{6}{2\dfrac{1-\lambda}{C} - 1}$ (> 0) outside these curves;
- on the other part of curve a_2 . . . $\theta_1 > 0$, $\theta_2 > 0$ $(\theta_1 = \theta_2)$;
- on the other part of curve b_2 . . . $\theta_1 > 0$, $\theta_2 = +\infty$;
- in the other part of the region. . . $\theta_1 > 0$, $\theta_2 > 0$.

for region III:

- at point A . . . . . $\theta_1 = -\infty$, $\theta_2 = -\infty$;
- at point D . . . . . $\theta_1 = -12$, $\theta_2 = -12$;
- on the other part of curve a_1 . . . $\theta_1 < -4$, $\theta_2 < -4$ $(\theta_1 = \theta_2)$;
- on the other part of curve b_1 . . . $\theta_1 = -\infty$, $\theta_2 < -4$;
- in the other part of the region. . . $\theta_1 < -4$, $\theta_2 < -4$.

for region IV:

- at point A $\theta_1 = +\infty$, $\theta_2 = -\infty$;
- at point B:
  - $\theta_1 = 0$, $\theta_2 = -\infty$ for curve b_2;
  - $\theta_1 = 0$, $\theta_2 = -6$ for curve c;
  - $\theta_1 = 0$, $\theta_2 = \dfrac{6}{2\dfrac{1-\lambda}{C} - 1}$ (< 0) outside these curves;
- on the other part of curve b_1 . . . $\theta_1 = +\infty$, $\theta_2 < -4$;
- on the other part of curve b_2 . . . $\theta_1 > 0$, $\theta_2 = -\infty$;
- on the other part of curve c. . . . $\theta_1 = 0$, $\theta_2 = -6$;
- in the other part of the region. . . $\theta_1 > 0$, $\theta_2 < -4$.

By application of formula (7), we find in consequence :

in region I : all the quantities $\varepsilon_1 - \varepsilon_4$ are complex (except on curve a);

in region II : all the quantities $\varepsilon_1 - \varepsilon_4$ are actual and we have : $0 \leq \varepsilon_1 \leq 1$, $0 \leq \varepsilon_2 \leq 1$, $\varepsilon_3 \geq 1$, $\varepsilon_4 \geq 1$;

in region III : all the quantities $\varepsilon_1 - \varepsilon_4$ are actual and we have : $\varepsilon_1 \cdot \quad 1$, $\varepsilon_2 \quad 1$, $\quad 1 \quad \varepsilon_3 \quad 0$, $\quad 1 \quad \varepsilon_4 \quad 0$;

in region IV : all the quantities $\varepsilon_1 - \varepsilon_4$ are actual and we have : $0 \quad \varepsilon_1 \quad 1$, $\varepsilon_2 \quad 1$, $\varepsilon_3 \quad 1$, $\quad 1 \quad \varepsilon_4 \quad 0$.

To provide for using actual quantities in region I, we here state :

$$\varepsilon_1 = e^{-\delta}(\cos\varphi - i\sin\varphi), \quad \varepsilon_2 = e^{-\delta}(\cos\varphi + i\sin\varphi) \ldots\ldots (14)$$

We then have :

$$\theta_{1,2} = \varepsilon_{1,2} - 2 + \frac{1}{\varepsilon_{1,2}} = 2(\operatorname{ch}\delta\cos\varphi - 1) \pm 2i\operatorname{sh}\delta\sin\varphi, \ldots (15)$$

and consequently,

$$\theta_1 + \theta_2 = 4(\operatorname{ch}\delta\cos\varphi - 1), \quad \theta_1\theta_2 = 4(\operatorname{ch}\delta - \cos\varphi)^2. \ldots\ldots (16)$$

Given that θ_1 and θ_2 are roots of the equation (5), we have :

$$\operatorname{ch}\delta\cos\varphi = 1 - \frac{1 - 8\lambda + \lambda^2}{4\left(\frac{1-\lambda^2}{C} - \lambda\right)}, \quad \operatorname{ch}\delta - \cos\varphi = \frac{1-\lambda}{2}\sqrt{\frac{6}{\frac{1-\lambda^2}{C} - \lambda}} \ldots (17)$$

Resolution of ch δ and cos φ of these equations gives, by calculation :

$$\operatorname{ch}\delta = \frac{\sqrt{8\frac{1-\lambda^2}{C} + (1+\lambda)^2} + (1-\lambda)\sqrt{3}}{2\sqrt{2}\sqrt{\frac{1-\lambda^2}{C} - \lambda}}, \ldots\ldots\ldots\ldots (18^a)$$

$$\cos\varphi = \frac{\sqrt{8\frac{1-\lambda^2}{C} + (1+\lambda)^2} - (1-\lambda)\sqrt{3}}{2\sqrt{2}\sqrt{\frac{1-\lambda^2}{C} - \lambda}} \quad (0 \leq \varphi \leq \pi). \ldots\ldots (18^b)$$

The following specifications can be readily derived :

$$\operatorname{ch}\delta = \frac{\sqrt{C+8} + \sqrt{3C}}{2\sqrt{2}}, \quad \cos\varphi = \frac{\sqrt{C+8} - \sqrt{3C}}{2\sqrt{2}} \text{ for } \lambda = 0, \ldots (19)$$

$$\operatorname{ch}\delta = \frac{5 - 4\lambda + 5\lambda^2}{1 - 8\lambda + \lambda^2}, \quad \varphi = \pi \text{ for curve } a; \ldots\ldots\ldots\ldots (20)$$

in the latter case C has been eliminated from (18^a) by the use of (9^a).

We may note here that one can multiply the general solutions of the equations (3, 7) and (3, 10) by any factor whatsoever. We shall use this fact in determining the solution which applies in the same way to equations (3, 8) and (3, 11), which we shall do in Chapter 5.

5. Solution of the equation for specific variations where a single force is exerted on the bar.

After having thus found the general solution of the equations (3, 7) and (3, 10), we shall seek the solution which also satisfies the equations (3, 8) and (3, 11). For this purpose, we shall first confine ourselves to the values of C and z which belong to region II, defined in Chapter 4. In this case, apart from the contour, we have :

$$|\varepsilon_1|<1,\ |\varepsilon_2|<1,\ |\varepsilon_3|>1 \text{ and } |\varepsilon_4|>1.$$

Since for $n=\pm\infty$ there should be $y_n=0$, we can express as a solution

$$\left.\begin{aligned} y_n &= A_1\varepsilon_1^n + A_2\varepsilon_2^n && \text{for } n\geq 1,\\ y_n &= B_1\varepsilon_3^n + B_2\varepsilon_4^n = B_1\varepsilon_1^{-n} + B_2\varepsilon_2^{-n} && \text{for } n\leq 0. \end{aligned}\right\} \quad (1)$$

From (4, 1), the first equation (4, 4) and (4, 7) we have at the same time :

$$\left.\begin{aligned} \frac{\nu M_n}{l} &= \frac{\theta A_1\varepsilon_1^n}{6+\theta_1} + \frac{\theta_2 A_2\varepsilon_2^n}{6+\theta_2} && \text{for } n\geq 1,\\ \frac{\nu M_n}{l} &= \frac{\theta_1 B_1\varepsilon_1^{-n}}{6+\theta_1} + \frac{\theta_2 B_2\varepsilon_2^{-n}}{6+\theta_2} && \text{for } n\leq 0. \end{aligned}\right\} \quad (2)$$

From this we get :

$$\left.\begin{aligned} &y_0 = B_1 + B_2, \quad y_1 = A_1\varepsilon_1 + A_2\varepsilon_2,\\ &\frac{\nu M_0}{l} = \frac{\theta_1 B_1}{6+\theta_1} + \frac{\theta_2 B_2}{6+\theta_2}, \quad \frac{\nu M_1}{l} = \frac{\theta_1 A_1\varepsilon_1}{6+\theta_1} + \frac{\theta_2 A_2\varepsilon_2}{6+\theta_2},\\ &\delta^2 y_0 = A_1\varepsilon_1 + A_2\varepsilon_2 + B_1(\varepsilon_1 - 2) + B_2(\varepsilon_2 - 2),\\ &\delta^2 y_1 = A_1\varepsilon_1(\varepsilon_1 - 2) + A_2\varepsilon_2(\varepsilon_2 - 2) + B_1 + B_2,\\ &\frac{\nu\delta^2 M_0}{l} = \frac{\theta_1 A_1\varepsilon_1}{6+\theta_1} + \frac{\theta_2 A_2\varepsilon_2}{6+\theta_2} + \frac{\theta_1 B_1(\varepsilon_1 - 2)}{6+\theta_1} + \frac{\theta_2 B_2(\varepsilon_2 - 2)}{6+\theta_2},\\ &\frac{\nu\delta^2 M_1}{l} = \frac{\theta_1 A_1\varepsilon_1(\varepsilon_1 - 2)}{6+\theta_1} + \frac{\theta_2 A_2\varepsilon_2(\varepsilon_2 - 2)}{6+\theta_2} + \frac{\theta_1 B_1}{6+\theta_1} + \frac{\theta_2 B_2}{6+\theta_2}. \end{aligned}\right\} \quad (3)$$

Substituting these values in the formulae (**3**, 8) and (**3**, 11), we have four equations for the constants A_1, A_2, B_1 and B_2. After evaluation, in which we use the relationships, which can be readily obtained from (**4**, 2), (**4**, 4) and (**4**, 6) we have :

$$\left.\begin{aligned}
&\frac{\theta_j}{6+\theta_j} - 1 = -\frac{6\varepsilon_j}{1+4\varepsilon_j+\varepsilon_j{}^2},\\
&\frac{\theta_j}{6+\theta_j}(\varepsilon_j+4) - (\varepsilon_j-2) = \frac{6}{1+4\varepsilon_j+\varepsilon_j{}^2},\\
&\frac{1-\lambda^2}{C}\frac{\theta_j}{6+\theta_j} - \lambda = -\frac{(1-\lambda)^2}{\theta_j},\\
&\frac{\varepsilon_j}{\theta_j}\left(\frac{\varepsilon_j}{1-\varepsilon_j}\right)^2 - 1 = \frac{\varepsilon_j-2}{\theta_j} - \frac{1}{(1-\varepsilon_j)^2},
\end{aligned}\right\} \quad (j=1,2) \quad \ldots\ldots\ldots \quad (4)$$

we find for these equations :

$$\left.\begin{aligned}
&\frac{\varepsilon_1(A_1-B_1)}{1+4\varepsilon_1+\varepsilon_1{}^2} + \frac{\varepsilon_2(A_2-B_2)}{1+4\varepsilon_2+\varepsilon_2{}^2} = \frac{\nu P\xi_p(1-\xi_p{}^2)}{6},\\
&\frac{-\varepsilon_1{}^2A_1+B_1}{1+4\varepsilon_1+\varepsilon_1{}^2} + \frac{-\varepsilon_2{}^2A_2+B_2}{1+4\varepsilon_2+\varepsilon_2{}^2} = \frac{\nu P\xi_p(1-\xi_p)(2-\xi_p)}{6},\\
&\frac{\varepsilon_1(A_1-B_1)}{(1-\varepsilon_1)^2} + \frac{\varepsilon_2(A_2-B_2)}{(1-\varepsilon_2)^2} = \frac{1+\lambda}{1-\lambda}\frac{\nu P\xi_p}{C},\\
&\frac{-\varepsilon_1{}^2A_1+B_1}{(1-\varepsilon_1)^2} + \frac{-\varepsilon_2{}^2A_2+B_2}{(1-\varepsilon_2)^2} = \frac{1+\lambda}{1-\lambda}\frac{\nu P(1-\xi_p)}{C}.
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (5)$$

Substituting in these equations the auxiliary unknowns :

$$X_j = -\varepsilon_j{}^2A_j + B_j,\ Y_j = A_j - B_j \quad (j=1,2) \quad \ldots\ldots \quad (6)$$

we can in the first place easily determine these auxiliary unknowns. Thence, the actual unknowns are derived from :

$$A_j(1-\varepsilon_j{}^2) = X_j + Y_j,\ B_j(1-\varepsilon_j{}^2) = X_j + \varepsilon_j{}^2Y_j \quad (j=1,2). \quad \ldots \quad (7)$$

We shall only indicate the result of the calculations in which the following have been used :

$$\left.\begin{aligned}
&\frac{1+4\varepsilon_2+\varepsilon_2^2}{1+4\varepsilon_1+\varepsilon_1^2}-\left(\frac{1-\varepsilon_2}{1-\varepsilon_1}\right)^2=-\frac{6(\varepsilon_1-\varepsilon_2)(1-\varepsilon_1\varepsilon_2)}{(1+4\varepsilon_1+\varepsilon_1^2)(1-\varepsilon_1)^2}\cdot\\
&\frac{1+4\varepsilon_1+\varepsilon_1^2}{1+4\varepsilon_2+\varepsilon_2^2}-\left(\frac{1-\varepsilon_1}{1-\varepsilon_2}\right)^2=-\frac{6(\varepsilon_1-\varepsilon_2)(1-\varepsilon_1\varepsilon_2)}{(1+4\varepsilon_2+\varepsilon_2^2)(1-\varepsilon_2)^2}\cdot
\end{aligned}\right\} \qquad (8)$$

Finally we find :

$$\left.\begin{aligned}
&\frac{36(\varepsilon_1-\varepsilon_2)(1-\varepsilon_1\varepsilon_2)}{\nu P(1+4\varepsilon_1+\varepsilon_1^2)}\frac{1+\varepsilon_1}{1-\varepsilon_1}A_1=-\xi_p(1-\xi_p)\left(2-\xi_p+\frac{1+\xi_p}{\varepsilon_1}\right)(1+4\varepsilon_2+\varepsilon_2^2)\\
&\qquad\qquad -\frac{6}{C}\frac{1+\lambda}{1-\lambda}\left(1-\xi_p+\frac{\xi_p}{\varepsilon_1}\right)(1-\varepsilon_2)^2,\\
&\frac{36(\varepsilon_1-\varepsilon_2)(1-\varepsilon_1\varepsilon_2)}{\nu P(1+4\varepsilon_2+\varepsilon_2^2)}\frac{1+\varepsilon_2}{1-\varepsilon_2}A_2=\xi_p(1-\xi_p)\left(2-\xi_p+\frac{1+\xi_p}{\varepsilon_2}\right)(1+4\varepsilon_1+\varepsilon_1^2)\\
&\qquad\qquad -\frac{6}{C}\frac{1+\lambda}{1-\lambda}\left(1-\xi_p+\frac{\xi_p}{\varepsilon_2}\right)(1-\varepsilon_1)^2,\\
&\frac{36(\varepsilon_1-\varepsilon_2)(1-\varepsilon_1\varepsilon_2)}{\nu P(1+4\varepsilon_1+\varepsilon_1^2)}\frac{1+\varepsilon_1}{1-\varepsilon_1}B_1=-\xi_p(1-\xi_p)\{2-\xi_p+(1+\xi_p)\varepsilon_1\}(1+4\varepsilon_2+\varepsilon_2^2)\\
&\qquad\qquad -\frac{6}{C}\frac{1+\lambda}{1-\lambda}(1-\xi_p+\xi_p\varepsilon_1)(1-\varepsilon_2)^2,\\
&\frac{36(\varepsilon_1-\varepsilon_2)(1-\varepsilon_1\varepsilon_2)}{\nu P(1+4\varepsilon_2+\varepsilon_2^2)}\frac{1+\varepsilon_2}{1-\varepsilon_2}B_2=\xi_p(1-\xi_p)\{2-\xi_p+(1+\xi_p)\varepsilon_2\}(1+4\varepsilon_1+\varepsilon_1^2)\\
&\qquad\qquad -\frac{6}{C}\frac{1+\lambda}{1-\lambda}(1-\xi_p+\xi_p\varepsilon_2)(1-\varepsilon_1)^2.
\end{aligned}\right\} \qquad (9)$$

Given this, we are ready to establish the formulae for y_n and M_n. The substitution of (9) for (1) gives for y_n

$$
\left.
\begin{aligned}
&36\,(\varepsilon_1-\varepsilon_2)\,(1-\varepsilon_1\varepsilon_2)\,\frac{y_n}{\nu P} = -\,(1+4\varepsilon_1+\varepsilon_1^2)\,(1+4\varepsilon_2+\varepsilon_2^2)\xi_p(1-\xi_p)\\
&\left[\frac{1-\varepsilon_1}{1+\varepsilon_1}\{(2-\xi_p)\varepsilon_1^{\,n}+(1+\xi_p)\varepsilon_1^{\,n-1}\}-\frac{1-\varepsilon_2}{1+\varepsilon_2}\{(2-\xi_p)\varepsilon_2^{\,n}+(1+\xi_p)\varepsilon_2^{\,n-1}\}\right]\\
&+\frac{6}{C}\,\frac{1+\lambda}{1-\lambda}\,(1-\varepsilon_1)\,(1-\varepsilon_2)\left[(1+4\varepsilon_1+\varepsilon_1^2)\,\frac{1-\varepsilon_2}{1+\varepsilon_1}\{(1-\xi_p)\varepsilon_1^{\,n}+\xi_p\varepsilon_1^{\,n-1}\}\right.\\
&\left.-(1+4\varepsilon_2+\varepsilon_2^2)\,\frac{1-\varepsilon_1}{1+\varepsilon_2}\{(1-\xi_p)\varepsilon_2^{\,n}+\xi_p\varepsilon_2^{\,n-1}\}\right] \qquad \text{for } n\geq 1,\\
&36\,(\varepsilon_1-\varepsilon_2)\,(1-\varepsilon_1\varepsilon_2)\,\frac{y_n}{\nu P} = -\,(1+4\varepsilon_1+\varepsilon_1^2)\,(1+4\varepsilon_1+\varepsilon_2^2)\xi_p(1-\xi_p)\\
&\left[\frac{1-\varepsilon_1}{1+\varepsilon_1}\{(2-\xi_p)\varepsilon_1^{\,-n}+(1+\xi_p)\,\varepsilon_1^{\,1-n}\}-\frac{1-\varepsilon_2}{1+\varepsilon_2}\{(2-\xi_p)\varepsilon_2^{\,-n}+(1+\xi_p)\varepsilon_2^{\,1-n}\}\right]\\
&+\frac{6}{C}\,\frac{1+\lambda}{1-\lambda}\,(1-\varepsilon_1)\,(1-\varepsilon_2)\left[(1+4\varepsilon_1+\varepsilon_1^2)\,\frac{1-\varepsilon_2}{1+\varepsilon_1}\{(1-\xi_p)\varepsilon_1^{\,-n}+\xi_p\varepsilon_1^{\,1-n}\}\right.\\
&\left.-(1+4\varepsilon_2+\varepsilon_2^2)\,\frac{1-\varepsilon_1}{1+\varepsilon_2}\{(1-\xi_p)\varepsilon_2^{\,-n}+\xi_p\varepsilon_2^{\,1-n}\}\right] \qquad \text{for } n\leq 0.
\end{aligned}
\right\} \quad (10)
$$

In addition, we get for the bending moment from (2) and (9)

$$
\begin{aligned}
&36\,(\varepsilon_1-\varepsilon_2)\,(1-\varepsilon_1\varepsilon_2)\,\frac{M_n}{Pl} =\\
&=-\,\xi_p(1-\xi_p)\left[(1+4\varepsilon_2+\varepsilon_2^2)\,\frac{(1-\varepsilon_1)^3}{1+\varepsilon_1}\{(2-\xi_p)\varepsilon_1^{\,n}+(1+\xi_p)\varepsilon_1^{\,n-1}\}\right.\\
&\left.-(1+4\varepsilon_1+\varepsilon_1^2)\,\frac{(1-\varepsilon_2)^3}{1+\varepsilon_2}\{(2-\xi_p)\varepsilon_2^{\,n}+(1+\xi_p)\varepsilon_2^{\,n-1}\}\right]\\
&-\frac{6}{C}\,\frac{1+\lambda}{1-\lambda}\,(1-\varepsilon_1)^2\,(1-\varepsilon_2)^2\left[\frac{1-\varepsilon_1}{1+\varepsilon_1}\{(1-\xi_p)\varepsilon_1^{\,n}+\xi_p\,\varepsilon_1^{\,n-1}\}\right.\\
&\left.-\frac{1-\varepsilon_2}{1+\varepsilon_2}\{(1-\xi_p)\varepsilon_2^{\,n}+\xi_p\,\varepsilon_2^{\,n-1}\}\right] \qquad \text{for } n\geq 1,
\end{aligned}
$$

$$36\,(\varepsilon_1 - \varepsilon_2)\,(1 - \varepsilon_1\varepsilon_2)\,\frac{M_n}{Pl} =$$

$$= -\,\xi_p(1-\xi_p)\left[(1 + 4\varepsilon_2 + \varepsilon_2^2)\,\frac{(1-\varepsilon_1)^3}{1+\varepsilon_1}\,\{(2-\xi_p)\varepsilon_1^{-n} + (1+\xi_p)\varepsilon_1^{1-n}\}\right.$$

$$\left. -\,(1 + 4\varepsilon_1 + \varepsilon_1^2)\,\frac{(1-\varepsilon_2)^3}{1+\varepsilon_2}\,\{(2-\xi_p)\varepsilon_2^{-n} + (1+\xi_p)\varepsilon_2^{1-n}\}\right]$$

$$+\,\frac{6}{C}\,\frac{1+\lambda}{1-\lambda}\,(1-\varepsilon_1)^2\,(1-\varepsilon_2)^2\left[\frac{1-\varepsilon_1}{1+\varepsilon_1}\,\{(1-\xi_p)\varepsilon_1^{-n} + \xi_p\,\varepsilon_1^{1-n}\}\right.$$

$$\left. -\,\frac{1-\varepsilon_2}{1+\varepsilon_2}\,\{(1-\xi_p)\varepsilon_2^{-n} + \xi_p\varepsilon_2^{1-n}\}\right] \qquad \text{for } n \leq 0.$$

In the following, we shall keep to the two most important cases: that in which the force P is located exactly above a bearing ($\xi_p = 0$[1]) and that in which the force is applied exactly mid-way between two bearings ($\xi_p = \frac{1}{2}$). For these cases, formulae (10) and (11) take the following simplified forms :

$$\left.\begin{aligned}
&6\,(\varepsilon_1 - \varepsilon_2)\,(1 - \varepsilon_1\varepsilon_2)\,\frac{y_n}{\nu P} = \frac{1}{C}\,\frac{1+\lambda}{1-\lambda}\,(1-\varepsilon_1)\,(1-\varepsilon_2)\\
&\left\{(1 + 4\varepsilon_1 + \varepsilon_1^2)\,\frac{1-\varepsilon_2}{1+\varepsilon_1}\,\varepsilon_1^n - (1 + 4\varepsilon_2 + \varepsilon_2^2)\,\frac{1-\varepsilon_1}{1+\varepsilon_2}\,\varepsilon_2^n\right\},\\
&6\,(\varepsilon_1 - \varepsilon_2)\,(1 - \varepsilon_1\varepsilon_2)\,\frac{M_n}{Pl}\\
&= \frac{1}{C}\,\frac{1+\lambda}{1-\lambda}\,(1-\varepsilon_1)^2\,(1-\varepsilon_2)^2\left(\frac{1-\varepsilon_1}{1+\varepsilon_1}\,\varepsilon_1^n - \frac{1-\varepsilon_2}{1+\varepsilon_2}\,\varepsilon_2^n\right),
\end{aligned}\right\} \quad \text{for } \xi_p = 0 \text{ and } n \geq 0, \qquad (12)$$

(1) In article 1 we used for this case the notation $\xi_p = 1$; however the notation $\xi_p = 0$ offers certain advantages.

$$12\,(\varepsilon_1 - \varepsilon_2)\,(1 - \varepsilon_1\varepsilon_2)\,\frac{y_n}{\nu P} =$$

$$= -\frac{1}{8}\,(1 + 4\varepsilon_1 + \varepsilon_1^2)\,(1 + 4\varepsilon_2 + \varepsilon_2^2)\left\{(1 - \varepsilon_1)\varepsilon_1^{n-1} - (1 - \varepsilon_2)\varepsilon_2^{n-1}\right\}$$

$$+ \frac{1}{C}\,\frac{1 + \lambda}{1 - \lambda}\,(1 - \varepsilon_1)\,(1 - \varepsilon_2)\left\{(1 + 4\varepsilon_1 + \varepsilon_1^2)\,(1 - \varepsilon_2)\varepsilon_1^{n-1}\right.$$

$$\left. - (1 + 4\varepsilon_2 + \varepsilon_2^2)\,(1 - \varepsilon_1)\varepsilon_2^{n-1}\right\},$$

$$12\,(\varepsilon_1 - \varepsilon_2)\,(1 - \varepsilon_1\varepsilon_2)\,\frac{M_n}{Pl} =$$

$$= -\frac{1}{8}\left\{(1 + 4\varepsilon_2 + \varepsilon_2^2)\,(1 - \varepsilon_1)^3\varepsilon_1^{n-1} - (1 + 4\varepsilon_1 + \varepsilon_1^2)\,(1 - \varepsilon_2)^3\varepsilon_2^{n-1}\right\}$$

$$+ \frac{1}{C}\,\frac{1 + \lambda}{1 - \lambda}\,(1 - \varepsilon_1)^2\,(1 - \varepsilon_2)^2\left\{(1 - \varepsilon_1)\varepsilon_1^{n-1} - (1 - \varepsilon_2)\varepsilon_2^{n-1}\right\}.$$

for $\xi_p = 1/2$ and $n \geq 1$ (13)

We have only indicated the formulae for $n \geq 0$ ($\xi_p = 0$) and for $n \geq 1$ ($\xi_p = 1/2$); from these can be derived the corresponding formulae for other values of n, substituting in the right-hand expression n for $-n$ (for $\xi_p = 0$) and n for $1 - n$ (for $\xi_p = 1/2$) respectively.

We may recall here the remark at the beginning of the Chapter, that we shall confine ourselves in the first place to values of C and λ which relate to region II indicated in Chapter **4.** For other values of these quantities, it is necessary to modify slightly the formulae which we have just derived. Before considering these we will, in the following Chapter, derive the formulae for bend and bending moment at the point of application of the force.

6. Bend and bending moment at the point of application of the force.

When the force is located exactly above a bearing, the bend y_p and the moment M_p are determined by :

$$y_p = y_o, \qquad M_p = M_o \qquad \text{for } \xi_p = 0, \quad \ldots\ldots\ldots \quad (1)$$

in which y_o and M_o are defined by (5, 12). We find :

$$\left.\begin{aligned} 3(1-\varepsilon_1\varepsilon_2)\frac{y_p}{\nu P} &= \frac{1}{C}\,\frac{1+\lambda}{1-\lambda}\,\frac{1-\varepsilon_1}{1+\varepsilon_1}\,\frac{1-\varepsilon_2}{1+\varepsilon_2}(2+\varepsilon_1+\varepsilon_2+2\varepsilon_1\varepsilon_2), \\ 3(1-\varepsilon_1\varepsilon_2)\frac{M_p}{Pl} &= -\frac{1}{C}\,\frac{1+\lambda}{1-\lambda}\,\frac{(1-\varepsilon_1)^2}{1-\varepsilon_1}\,\frac{(1-\varepsilon_2)^2}{1-\varepsilon_2}, \end{aligned}\right\} \text{for } \xi_p = 0. \quad \ldots\ldots (2)$$

The case $\xi_p = \frac{1}{2}$ is more complicated. We can then use the quantities y_p and M_p as follows. Let us imagine the bar cut outside the span 01 as nearly as possible to the bearing 0 and outside the same span as nearly as possible to bearing 1, and consider the portion of the bar between these two points, when it is loaded with the force P and the forces and internal moments which exist in the uncut bar in the

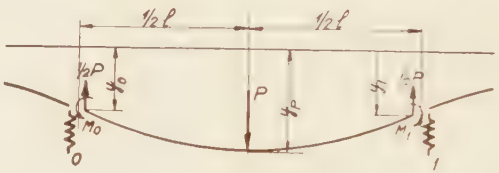


Fig. 4. — Portion of the bar between bearings 0 and 1 with the forces and moments loading it, for $\xi_p = 1/2$.

same places (fig. 4). Because of the symmetry of the bar in relation to the line of application of the force, we have $y_1 = y_o$, $M_1 = M_o$ and since the angular deviation of the bar at the point P is equal to zero we have :

$$y_p - y_o = \frac{Pl^3}{48\,EI} - \frac{M_o l^2}{8\,EI},$$

or by introducing the quantity ν (3, 6)

$$\frac{y_p}{\nu P} = \frac{y_o}{\nu P} - \frac{3}{4}\frac{M_o}{Pl} + \frac{1}{8} \quad \text{for } \xi_p = \frac{1}{2}. \quad \ldots\ldots\ldots (3)$$

It can therefore be concluded from the equilibrium of each of the two parts of the bar, that the bending moment at the point of application of P is equal to :

$$M_p = \frac{1}{4}Pl - M_o$$

or

$$\frac{M_p}{Pl} = \frac{M_o}{Pl} - \frac{1}{4} \quad \text{for } \xi_p = \frac{1}{2}. \quad \ldots\ldots\ldots (4)$$

With the help of the formulae (**5, 13**), we find by calculation :

$$\left.\begin{aligned}
&48\,(1-\varepsilon_1\varepsilon_2)\,\frac{y_p}{\nu P} = \\
&= \frac{1}{8}\Big\{31 + 22\,(\varepsilon_1+\varepsilon_2) + \varepsilon_1^2 + 58\varepsilon_1\varepsilon_2 + \varepsilon_2^2 + 4\varepsilon_1\varepsilon_2\,(\varepsilon_1+\varepsilon_2) + \varepsilon_1^2\varepsilon_2^2\Big\} \\
&+ \frac{1}{C}\,\frac{1+\lambda}{1-\lambda}\,(1-\varepsilon_1)\,(1-\varepsilon_2)\,(23+\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2), \\
&12\,(1-\varepsilon_1\varepsilon_2)\,\frac{M_p}{Pl} = \\
&= -\frac{1}{8}\Big\{17 + 2\,(\varepsilon_1+\varepsilon_2) - \varepsilon_1^2 - 10\varepsilon_1\varepsilon_2 - \varepsilon_2^2 - 4\varepsilon_1\varepsilon_2\,(\varepsilon_1+\varepsilon_2) - \varepsilon_1^2\varepsilon_2^2\Big\} \\
&- \frac{1}{C}\,\frac{1+\lambda}{1-\lambda}\,(1-\varepsilon_1)^2\,(1-\varepsilon_2)^2,
\end{aligned}\right\} \quad \text{for } \xi_p = \frac{1}{2}. \qquad (5)$$

The question arises whether, for a value of ξ other than the values considered here : $\left(\xi_p = 0 \text{ and } \xi_p = \frac{1}{2}\right)$ the bend and the moment at the point of application are greater. We shall not undertake an examination of this question, in view of the difficulties of the calculations involved.

7. Formulae for bend and bending moment for complex values of ε_1 and ε_2.

When the values of C and λ are such that the junction point (C, λ) in figure 3 is located in region III or IV, formulae (**5, 12**), (**5, 13**), (**6, 2**) and (**6, 5**) remain valid if we replace only :

ε_2 by ε_4 in region III

and ε_2 by ε_4 and also ε_1 by ε_3 in region IV,

in accordance with the remark in Chapter **4** on the size of the constants $\varepsilon_1 - \varepsilon_4$.

When the point (C, λ) is in region I, ε_1 and ε_2 are complex. In this case, we can introduce into the formulae mentioned above the quantities φ and δ from (**4, 14**) :

$$\varepsilon_1 = e^{-\delta}\,(\cos\varphi - i\sin\varphi), \qquad \varepsilon_2 = e^{-\delta}\,(\cos\varphi + i\sin\varphi). \quad \ldots\ldots \quad (1)$$

In addition, we can write :

$$\left.\begin{aligned}
&\frac{1}{\varepsilon_1} = e^{\delta}\,(\cos\varphi + i\sin\varphi), \qquad \frac{1}{\varepsilon_2} = e^{\delta}\,(\cos\varphi - i\sin\varphi), \\
&\varepsilon_1 - \varepsilon_2 = -2\,i\,e^{-\delta}\,\sin\varphi \qquad \varepsilon_1 + \varepsilon_2 = 2\,e^{-\delta}\,\cos\varphi, \\
&\varepsilon_1\varepsilon_2 = e^{-2\delta}, \qquad 1 - \varepsilon_1\varepsilon_2 = 2\,e^{-\delta}\,\mathrm{sh}\,\delta, \\
&1 + 4\varepsilon_1 + \varepsilon_1^2 = 2\,(\mathrm{ch}\,\delta\cos\varphi + 2 + i\,\mathrm{sh}\,\delta\sin\varphi)\varepsilon_1, \\
&1 + 4\varepsilon_2 + \varepsilon_2^2 = 2\,(\mathrm{ch}\,\delta\cos\varphi + 2 - i\,\mathrm{sh}\,\delta\sin\varphi)\varepsilon_2,
\end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

and at the same time, from the theory of DE MOIVRE

$$\varepsilon_1{}^n = e^{-n\delta}\ (\cos\ n\ \varphi - i\ \sin\ n\ \varphi), \qquad \varepsilon_2{}^n = e^{-n\delta}\ (\cos\ n\ \varphi + i\ \sin\ n\ \varphi). \quad . \quad . \quad . \quad . \quad (3)$$

Thus formulae (**5**, 12) and (**5**, **13**) are converted to :

$$6\ \mathrm{sh}\ \delta\ \sin\ \varphi\ \frac{y_n}{\nu P} =$$

$$= \frac{1}{C}\frac{1+\lambda}{1-\lambda}\ e^{-n\delta}\frac{\mathrm{ch}\ \delta - \cos\varphi}{\mathrm{ch}\ \delta + \cos\varphi}\Big[(\mathrm{ch}\ \delta + \cos\varphi)\{e^{\delta}\sin\ (n+1)\ \varphi - e^{-\delta}\sin\ (n-1)\ \varphi\}$$

$$- \mathrm{sh}\ \delta\ \sin\ \varphi\{e^{\delta}\cos\ (n+1)\ \varphi - e^{-\delta}\cos\ (n-1)\ \varphi\}\Big],$$

$$3\ \mathrm{sh}\ \delta\ \sin\ \varphi\ \frac{M_n}{Pl} =$$

$$= \frac{1}{C}\frac{1+\lambda}{1-\lambda}\ e^{-n\delta}\frac{(\mathrm{ch}\ \delta - \cos\varphi)^2}{\mathrm{ch}\ \delta + \cos\varphi}\ (\mathrm{sh}\ \delta\ \sin\ n\varphi - \sin\ \varphi\ \cos\ n\varphi),$$

for $\xi_t = 0$ and $n \geq 0$, (4)

$$6\ \mathrm{sh}\ \delta\ \sin\ \varphi\ \frac{y_n}{\nu P} =$$

$$= -\frac{1}{8}e^{-(n-1)\delta}\{(\mathrm{ch}\ \delta - \cos\varphi)^2 + 3(2\ \mathrm{ch}\ \delta\cos\varphi + 1)\}\{\sin\ (n-1)\ \varphi - e^{-\delta}\sin\ n\varphi\}$$

$$+ \frac{1}{C}\frac{1+\lambda}{1-\lambda}\ e^{-(n-1)\delta}(\mathrm{ch}\ \delta - \cos\varphi)\Big[(\mathrm{ch}\ \delta\cos\varphi - 2)\sin n\varphi - e^{-\delta}\sin\ (n-1)\varphi\}$$

$$- \mathrm{sh}\ \delta\ \sin\ \varphi\{\cos\ n\varphi - e^{-\delta}\cos\ (n-1)\varphi\}\Big],$$

$$6\ \mathrm{sh}\ \delta\ \sin\ \varphi\ \frac{M_n}{Pl} = -\frac{1}{16}e^{-(n-1)\delta}\Big[(\mathrm{ch}\ \delta\cos\varphi + 2)$$

$$\{e^{\delta}\sin\ (n-2)\ \varphi - 3\ \sin\ (n-1)\ \varphi + 3e^{-\delta}\sin\ n\varphi - e^{-2\delta}\sin\ (n+1)\varphi\}$$

$$+ \mathrm{sh}\ \delta\ \sin\ \varphi\{e^{\delta}\cos\ (n-2)\ \varphi - 3\cos\ (n-1)\ \varphi + 3e^{-\delta}\cos\ n\varphi - e^{-2\delta}\cos\ (n+1)\varphi\}\Big]$$

$$+ \frac{1}{C}\frac{1+\lambda}{1-\lambda}e^{-(n-1)\delta}\ (\mathrm{ch}\ \delta - \cos\varphi)^2\{\sin\ (n-1)\ \varphi - e^{-\delta}\ \sin\ n\varphi\},$$

for $\xi_p = \frac{1}{2}$ and $n \geq 1$. (5)

We substitute the values (1) and (2) for ε_1, ε_2, etc. Also, in formulae (**6**, 2) and (**6**, 5) for the bends and bending moments at the point of application of the force and we find, respectively

$$\left.\begin{aligned} 3\ \mathrm{sh}\ \delta\ \frac{y_p}{\nu P} &= \frac{1}{C}\ \frac{1+\lambda}{1-\lambda}\ \frac{\mathrm{ch}\ \delta - \cos\varphi}{\mathrm{ch}\ \delta + \cos\varphi}\ (2\ \mathrm{ch}\ \delta + \cos\varphi), \\ 3\ \mathrm{sh}\ \delta\ \frac{M_p}{Pl} &= -\ \frac{1}{C}\ \frac{1+\lambda}{1-\lambda}\ \frac{(\mathrm{ch}\ \delta - \cos\varphi)^2}{\mathrm{ch}\ \delta + \cos\varphi}, \end{aligned}\right\} \text{ for } \xi_p = 0, \quad . \ . \ . \quad (6)$$

$$\left.\begin{aligned} 24\ \mathrm{sh}\ \delta\ \frac{y_p}{\nu P} &= \\ &= \frac{1}{16}\ e^{-\delta}\ (16\ \mathrm{ch}\ 2\delta + 15\ \mathrm{sh}\ 2\delta + 26\ \mathrm{ch}\ \delta \cos\varphi + 18\ \mathrm{sh}\ \delta \cos\varphi + \cos 2\varphi + 29) \\ &+ \frac{1}{C}\ \frac{1+\lambda}{1-\lambda}\ e^{-\delta}\ (\mathrm{ch}\ \delta - \cos\varphi)\ (11\ \mathrm{ch}\ \delta + 12\ \mathrm{sh}\ \delta + \cos\varphi), \\ 6\ \mathrm{sh}\ \delta\ \frac{M_p}{Pl} &= \\ &= -\ \frac{1}{16}\ e^{-\delta}\ (8\ \mathrm{ch}\ 2\delta + 9\ \mathrm{sh}\ 2\delta - 2\ \mathrm{ch}\ \delta \cos\varphi + 6\ \mathrm{sh}\ \delta \cos\varphi - \cos 2\varphi - 5) \\ &- \frac{1}{C}\ \frac{1+\lambda}{1-\lambda}\ e^{-\delta}\ (\mathrm{ch}\ \delta - \cos\varphi)^2, \end{aligned}\right\} \text{ for } \xi_p = \frac{1}{2}. \quad (7)$$

On curve a in the plane (C, λ) (see fig. 3), we get from (**4**, 20) $\varphi = \pi$. As we can easily see from the rule of DE L'HÔPITAL, the formulae (4) – (7) concern respectively the simplified expressions :

$$\left.\begin{aligned} 3\ \mathrm{sh}\delta\ \frac{y_n}{\nu P} &= \\ &- \frac{(-1)^n}{C}\ \frac{1+\lambda}{1-\lambda}\ e^{-n\delta}\ \frac{\mathrm{ch}\ \delta + 1}{\mathrm{ch}\ \delta - 1}\left[(\mathrm{ch}\ \delta - 2)\{(n+1)\ \mathrm{sh}\ \delta + e^{-\delta}\} - \mathrm{sh}^2\delta\right], \\ 3\ \mathrm{sh}\ \delta\ \frac{M_n}{Pl} &= -\ \frac{(-1)^n}{C}\ \frac{1+\lambda}{1-\lambda}\ e^{-n\delta}\ \frac{(\mathrm{ch}\ \delta + 1)^2}{\mathrm{ch}\ \delta - 1}\ (n\ \mathrm{sh}\ \delta + 1), \end{aligned}\right\} \text{ for } \xi_p = 0 \text{ and } n \geq 0, \quad (8)$$

$$6\ \mathrm{sh}\ \delta\ \frac{y_n}{\nu P} = \frac{(-1)^{n-1}}{8} e^{-(n-1)\delta}\ (\mathrm{ch}\ \delta - 2)^2\ (n - 1 + ne^{-\delta})$$

$$- \frac{(-1)^{n-1}}{C}\ \frac{1+\lambda}{1-\lambda} e^{-(n-1)\delta} (\mathrm{ch}\delta + 1)\Big[(\mathrm{ch}\delta - 2)\{n + (n-1)e^{-\delta}\} - \mathrm{sh}\delta\ (1 + e^{-\delta})\Big],$$

$$6\ \mathrm{sh}\ \delta\ \frac{M_n}{Pl} =$$

$$= \frac{(-1)^{n-1}}{16} e^{-(n-1)\delta}\Big[(\mathrm{ch}\ \delta - 2)\{(n-2)\ e^{\delta} - 3\ (n-1) - 3ne^{-\delta} + (n+1)\ e^{-2\delta}\}$$

$$+ \mathrm{sh}\ \delta\ (e^{\delta} + 3 + 3e^{-\delta} + e^{-2\delta})\Big]$$

$$- \frac{(-1)^{n-1}}{C}\ \frac{1+\lambda}{1-\lambda} e^{-(n-1)\delta}\ (\mathrm{ch}\ \delta + 1)^2\ (n - 1 + ne^{-\delta}),$$

for $\xi_p = \frac{1}{2}$ and $n \geq 1$ (9)

$$3\ \mathrm{sh}\ \delta\ \frac{y_p}{\nu P} = \frac{1}{C}\ \frac{1+\lambda}{1-\lambda}\ \frac{\mathrm{ch}\ \delta + 1}{\mathrm{ch}\ \delta - 1}\ (2\ \mathrm{ch}\ \delta - 1),$$

$$3\ \mathrm{sh}\ \delta\ \frac{M_p}{Pl} = -\frac{1}{C}\ \frac{1+\lambda}{1-\lambda}\ \frac{(\mathrm{ch}\ \delta + 1)^2}{\mathrm{ch}\ \delta - 1},$$

for $\xi_p = 0$, ... (10)

$$24\ \mathrm{sh}\ \delta\ \frac{y_p}{\nu P} = \frac{1}{16} e^{-\delta}\ (16\ \mathrm{ch}\ 2\delta + 15\ \mathrm{sh}\ 2\delta - 26\ \mathrm{ch}\ \delta - 18\ \mathrm{sh}\ \delta + 30)$$

$$+ \frac{1}{C}\ \frac{1+\lambda}{1-\lambda} e^{-\delta}\ (\mathrm{ch}\ \delta + 1)\ (11\ \mathrm{ch}\ \delta + 12\ \mathrm{sh}\ \delta - 1),$$

$$6\ \mathrm{sh}\ \delta\ \frac{M_p}{Pl} = -\frac{1}{16} e^{-\delta}\ (8\ \mathrm{ch}\ 2\ \delta + 9\ \mathrm{sh}\ 2\delta + 2\mathrm{ch}\delta - 6\ \mathrm{sh}\ \delta - 6)$$

$$- \frac{1}{C}\ \frac{1+\lambda}{1-\lambda} e^{-\delta}\ (\mathrm{ch}\ \delta + 1)^2,$$

for $\xi_p = \frac{1}{2}$. (11)

For the point A located on curve *a*, we have $\lambda = \lambda_1 = 4 = \sqrt{15}$, $C = C_1 = 2\sqrt{15}$ and $\delta = \infty$ (see Chapter 4). Substituting this in the formulae (8) — (11) we find :

$$\left.\begin{aligned}
&\frac{y_n}{\nu P} = 0 \text{ for } n = 2,\ 3 \ldots \infty,\ -2,\ -3 \ldots -\infty,\\
&\frac{y_o}{\nu P} = \frac{y_p}{\nu P} = \frac{1}{9},\quad \frac{y_1}{\nu P} = \frac{y_{-1}}{\nu P} = \frac{1}{36},\\
&\frac{M_n}{Pl} = 0 \text{ for } n = 2,\ 3 \ldots \infty,\ -2,\ -3 \ldots -\infty,\\
&\frac{M_o}{Pl} = \frac{M_p}{Pl} = -\frac{1}{18},\quad \frac{M_1}{Pl} = \frac{M_{-1}}{Pl} = \frac{1}{36},
\end{aligned}\right\} \text{ for } \xi_p = 0, \quad (12)$$

$$\left.\begin{aligned}
&\frac{y_n}{\nu P} = 0 \text{ for } n = 3,\ 4 \ldots \infty, -2,\ -3 \ldots -\infty,\\
&\frac{y_1}{\nu P} = \frac{y_o}{\nu P} = \frac{23}{288},\quad \frac{y_2}{\nu P} = \frac{y_{-1}}{\nu P} = \frac{1}{288},\quad \frac{y_p}{\nu P} = \frac{185}{1152},\\
&\frac{M_n}{Pl} = 0 \text{ for } n = 3,\ 4 \ldots \infty,\ -2,\ -3 \ldots -\infty,\\
&\frac{M_1}{Pl} = \frac{M_o}{Pl} = \frac{17}{288},\quad \frac{M_2}{Pl} = \frac{M_{-1}}{Pl} = \frac{1}{288},\quad \frac{M_p}{Pl} = -\frac{55}{288},
\end{aligned}\right\} \text{ for } \xi_p = \frac{1}{2}. \quad (13)$$

For these values of C and λ there is the curious fact that the bar, whilst of infinite length, is deformed only within a limited zone.

8. The continuous method.

After compiling the formulae relating to a bar resting on distinct bearings, we shall also consider here, as in Article 1, how such a calculation is undertaken when the bar is supported on a continuous bed. When a single concentrated force is applied to this bed, the bend must be produced in accordance with the depressions reproduced in figure 1 for a series of disconnected resilient bearings.

For this reason, it is evident that the bend must be represented by the formula :

$$\left.\begin{aligned}
y(x) &= \mu e^{-\nu(\bar{x}-x)} P \quad \text{for } \bar{x} > x,\\
y(x) &= \mu e^{-\nu(x-\bar{x})} P \quad \text{for } \bar{x} < x,
\end{aligned}\right\} \ldots\ldots\ldots (1)$$

in which $\bar{x}$ represents the location of the application of the force P, x indicates the measuring position of the bend (see fig. 5) and μ and ν are constants of elasticity which have, respectively, the dimensions [ν = length^{-1}] and [μ = length : force].

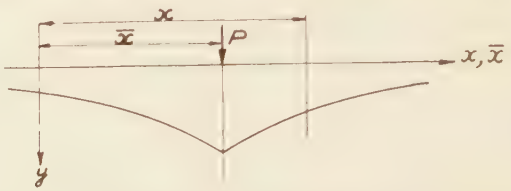


Fig. 5. — Deflection of a bed loaded with a single force.

In the first place, we shall have to consider what the relationship should be between the constants μ and ν, the quantities μ and λ introduced in Chapter 2 and the distance between the bearings l. At first sight, it seems necessary to determine this relationship in such a way that in loading the *bed* by a single force P_m, the depression of the bed at the same place $x = x = (m - n)l$, corresponds to the depression of the n^{th}. bearing, when the m^{th}. bearing is loaded only with a force P_m of the same size. This entails, for the satisfying of the relationship, from (1) and (2, 1) :

$$\mu = \bar{\mu}, \quad \lambda = e^{-\nu l}, \quad \nu = -\frac{ln\lambda}{l}. \qquad (2)$$

This, however, gives rise to the difficulty that, when it is desired to use the standard ZIMMERMANN theory, and for this purpose $\lambda \to 0$ is applied, the constant ν tends towards ∞.

For this reason, it is better to make comparisons, not between cases where the whole of the resilient bearings and the bed are loaded with a single force, but between cases where the whole range of bearings is loaded with a series of concentrated forces P_m ($m = ..., -2, -1, 1, 2, ...$) located above each bearing and the bed is loaded with a continuous charge $q(x)$. The relation between the deflections and the forces which apply to the whole range of bearings is indicated by (2, 4); it is not difficult to derive a corresponding relation for the case of a resilient bed. In effect, in place of (1), we usually apply in the latter case :

$$y(x) = \mu\left\{\int_{\bar{x}=-\infty}^{\bar{x}=x} e^{-\nu(x-\bar{x})}\, q(\bar{x})d\bar{x} + \int_{\bar{x}=x}^{\bar{x}=\infty} e^{-\nu(\bar{x}-x)}\, q(\bar{x})d\bar{x}\right\}. \qquad (3)$$

Differentiating this expression twice, we find :

$$\frac{dy}{dx} = \mu\nu\left\{-\int_{\bar{x}=-\infty}^{x=x} e^{-\nu(x-\bar{x})}q(\bar{x})d\bar{x} + \int_{\bar{x}=x}^{x=+\infty} e^{-\nu(\bar{x}-x)}\, q(\bar{x})d\bar{x}\right\},$$

$$\frac{d^2y}{dx^2} = \mu\nu^2\left\{\int_{x=-\infty}^{\bar{x}=x} e^{-\nu(x-\bar{x})}q(\bar{x})d\bar{x} + \int_{\bar{x}=x}^{\bar{x}=+\infty} e^{-\nu(\bar{x}-x)}q(\bar{x})d\bar{x}\right\} - 2\mu\nu q(x) \qquad (4)$$

Then eliminating the expression between the double curved brackets (3) and (4) gives the relationship between y and q

$$q(x) - \frac{\nu}{2\mu} y(x) = \frac{1}{2\mu\nu} \frac{d^2y(x)}{dx^2} \quad \ldots\ldots\ldots\ldots \quad (5)$$

Under this equation, we can put the relationship (**2**, 4) :

$$\frac{P_n}{l} = \frac{1-\lambda}{\bar{\mu}l(1+\lambda)} y_n - \frac{l\lambda}{\bar{\mu}(1-\lambda^2)} \frac{\delta^2 y_n}{l^2} \quad \ldots\ldots\ldots\ldots \quad (6)$$

If we now select the constants μ and ν so that at the position of each bearing :

$$q(x) = \frac{P_n}{l}, \quad y(x) = y_n \text{ and } \frac{d^2y(x)}{dx^2} = \frac{\Delta^2 y_n}{l^2}.$$

In this case, it is necessary that :

$$\frac{\nu}{2\mu} = \frac{1}{\bar{\mu}l} \frac{1-\lambda}{1+\lambda}, \quad \frac{1}{2\mu\nu} = \frac{l}{\bar{\mu}} \frac{\lambda}{1-\lambda^2}, \quad \ldots\ldots\ldots \quad (7)$$

from these we obtain by solution of μ and ν

$$\mu = \frac{\bar{\mu}(1+\lambda)}{2\sqrt{\lambda}}, \quad \nu = \frac{1-\lambda}{l\sqrt{\lambda}}. \quad \ldots\ldots\ldots\ldots \quad (8)$$

We have thus found expressions for the resilient constants μ and ν, in which we can without any disadvantage, let $\lambda = 0$.

We then derive the differential equation for the deflection of the bar. From the known expressions $EI \frac{d^2y}{dx^2} = M$, $\frac{d^2M}{dx^2} = q$ and formula (5), we obtain this equation

$$EI \frac{d^4y}{dx^4} + \frac{1}{2\mu\nu} \frac{d^2y}{dx^2} + \frac{\nu}{2\mu} y = 0,$$

or with

$$\alpha = \sqrt[4]{\frac{\nu}{8\mu EI}}, \quad \delta = \sqrt{\frac{1}{8\mu\nu EI}}, \quad \ldots\ldots\ldots\ldots \quad (9)$$

$$\frac{d^4y}{dx^4} + 4\delta^2 \frac{d^2y}{dx^2} + 4\alpha^4 y = 0. \quad \ldots\ldots\ldots\ldots \quad (10)$$

Taking, in (7) and (9) $\lambda = 0$, we find :

$$\alpha = \sqrt[4]{\frac{1}{4\,\bar{\mu} l EI}}, \quad \delta = 0,$$

which corresponds to equations (5, 2) and (5, 4) of Article 1. We will select a system of co-ordinates so that $x = 0$ at the point of application of the concentrated force P. As in Article 1, we must determine two solutions for (9), one solution (indicated by y) valid for $x \geq 0$ and one solution (y^*) valid for $x \leq 0$. The extreme conditions are now equally :

$$\left.\begin{array}{ll} y \text{ and its derivatives} = 0 & \text{for } x = +\infty, \\ y^* \text{ and its derivatives} = 0 & \text{for } x = -\infty, \\ y = y^*,\ y' = y^{*\prime},\ y'' = y^{*\prime\prime},\ y''' = y^{*\prime\prime\prime} + \dfrac{P}{EI} & \text{for } x = 0. \end{array}\right\} \quad . \ . \ . \ . \ (11)$$

The substitution of $y = e^{rx}$ in (10) gives the characteristic equation

$$r^4 + 4\,\delta^2 r^2 + 4\,\alpha^4 = 0; \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (12)$$

the solution of which is

$$r_1 = -r_3 = \sqrt{\delta^2 + \alpha^2} + \sqrt{\delta^2 - \alpha^2}, \quad r_2 = -r_4 = \sqrt{\delta^2 + \alpha^2} - \sqrt{\delta^2 - \alpha^2}. \quad (13)$$

For $\delta < \alpha$ all the values of r_j ($j = 1, \ldots 4$) are complex, for $\delta \geq \alpha$ they are actual. For $\delta = \alpha$ we have, from (9),

$$8\mu\nu^3 EI = 1. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (14)$$

TABLE II.

Some values of C and λ relating to curve e.

λ	C
0	∞
0.1	53.46
0.15	20.9258
0.2	10.24
0.25	5.625
0.3	3.3030
0.4	1.26
0.5	0.5
0.6	0.1896
0.7	0.0624
0.8	0.015
0.9	0.0016
1	0

Substituting the values (8) for μ and ν and in accordance with (3, 6) and (3, 9) taking $\frac{l^3}{6\,\bar{\mu} EI} = C$, we find, as equation of the curve in the plan (C, λ), for which $\alpha = \delta$.

$$C = \frac{2}{3}\frac{(1-\lambda^2)(1-\lambda)^2}{\lambda^2} \quad . \ . \ . \ . \ (15)$$

Table II gives several corresponding values of C and λ from (15), whilst the curve represented by (15) is shewn in figure 3 (curve e). In the part of the frame under consideration, of the plane (C, λ) from the side of the curve in

which it originates, we have $\delta \leq \alpha$, in the other part $\delta \geq \alpha$.

In view of the extreme conditions (11) for $x = \pm \infty$ and from the fact that the deflection must be symmetrical in x_1, the solution of (10) is :

$$\left.\begin{aligned} y &= B_1 e^{-r_1 x} + B_2 e^{-r_2 x} \quad \text{for } x \geq 0, \\ y^* &= B_1 e^{r_1 x} + B_2 e^{r_2 x} \quad \text{for } x \leq 0. \end{aligned}\right\} \quad \ldots\ldots\ldots (16)$$

The other extreme conditions give for the constants of integration :

$$B_1 = -\frac{P}{2r_1(r_1^2 - r_2^2)EI}, \quad B_2 = \frac{P}{2r_2(r_1^2 - r_2^2)EI}, \quad \ldots\ldots (17)$$

from which we can deduce the formula for the deflection of the bar :

$$y = P\frac{r_1 e^{-r_2 x} - r_2 e^{-r_1 x}}{2r_1 r_2(r_1^2 - r_2^2)EI} \quad \text{for } x \geq 0, \quad \ldots\ldots\ldots (18)$$

or after substitution of r_1 and r_2 (13)

$$y = Pe^{-x\sqrt{\alpha^2+\delta^2}} \frac{\sqrt{\alpha^2-\delta^2}\cos x\sqrt{\alpha^2-\delta^2} + \sqrt{\alpha^2+\delta^2}\sin x\sqrt{\alpha^2-\delta^2}}{8\alpha^2\sqrt{\alpha^4-\delta^4}EI}$$
$$\text{for } x \geq 0 \text{ and } \alpha > \delta, \quad \ldots\ldots (19a)$$

$$y = Pe^{-\alpha x\sqrt{2}} \frac{1+\alpha x\sqrt{2}}{8\alpha^3 EI\sqrt{2}} \quad \text{for } x \geq 0 \text{ and } \alpha = \delta, \quad \ldots\ldots (19b)$$

$$y = Pe^{-x\sqrt{\delta^2+\alpha^2}} \frac{\sqrt{\delta^2-\alpha^2}\,\mathrm{ch}\, x\sqrt{\delta^2-\alpha^2} + \sqrt{\delta^2+\alpha^2}\,\mathrm{sh}\, x\sqrt{\delta^2-\alpha^2}}{8\alpha^2\sqrt{\delta^4-\alpha^4}EI}$$
$$\text{for } x \geq 0 \text{ and } \alpha < \delta. \quad \ldots\ldots (19c)$$

The bending moment results from $M = EI\frac{d^2y}{dx^2}$ and is therefore equal to :

$$M = -Pe^{-x\sqrt{\alpha^2+\delta^2}} \frac{\sqrt{\alpha^2-\delta^2}\cos\sqrt{\alpha^2-\delta^2} - \sqrt{\alpha^2+\delta^2}\sin x\sqrt{\alpha^2-\delta^2}}{4\sqrt{\alpha^4-\delta^4}}$$
$$\text{for } x \geq 0 \text{ and } \alpha > \delta, \quad \ldots\ldots (20a)$$

$$M = -Pe^{-\alpha x\sqrt{2}} \frac{1-\alpha x\sqrt{2}}{4\alpha\sqrt{2}} \quad \text{for } x \geq 0 \text{ and } \alpha = \delta, \quad \ldots\ldots (20b)$$

$$M = -Pe^{-x\sqrt{\delta^2+\alpha^2}} \frac{\sqrt{\delta^2-\alpha^2}\,\mathrm{ch}\, x\sqrt{\delta^2-\alpha^2} - \sqrt{\delta^2+\alpha^2}\,\mathrm{sh}\, x\sqrt{\delta^2-\alpha^2}}{4\sqrt{\delta^4-\alpha^4}}$$
$$\text{for } x \geq 0 \text{ and } \alpha < \delta. \quad \ldots\ldots (20c)$$

These formulae are established in such a form that for the region in which they are valid, all the quantities emerging are actual. In deriving (19^b) and (20^b) we have made a limit conversion. From there, we have confined ourselves to $x \geqq 0$.

To provide easier comparison of these formulae with the relationships found with the bar resting on disconnected bearings, we shall first express α and δ in the constants C and λ used earlier and the distance between the bearings l. From (9), (8) and (3, 6) we get

$$\alpha l = \sqrt[4]{\frac{3C}{2}\frac{1-\lambda}{1+\lambda}}, \quad \delta l = \sqrt{\frac{3C}{2}\frac{\lambda}{1-\lambda^2}}; \quad \ldots\ldots\ldots (21)$$

we then take

$$\alpha l = a, \quad \delta l = d, \quad l\sqrt{\alpha^2 + \delta^2} = b_1, \quad l\sqrt{\alpha^2 - \delta^2} = b_2, \quad l\sqrt{\delta^2 - \alpha^2} = b_3,$$

and

$$\frac{x}{l} = \xi, \quad \ldots\ldots\ldots\ldots\ldots (22)$$

so that we have for the quantities a, d, b_1, b_2 and b_3

$$\left.\begin{aligned} a &= \sqrt[4]{\frac{3C}{2}\frac{1-\lambda}{1+\lambda}}, \quad d = \sqrt{\frac{3C}{2}\frac{\lambda}{1-\lambda^2}}, \\ b_1 &= \sqrt{(1-\lambda)\sqrt{\frac{3}{2}\frac{C}{1-\lambda^2}} + \lambda\frac{3}{2}\frac{C}{1-\lambda^2}}, \\ b_2 &= \sqrt{(1-\lambda)\sqrt{\frac{3}{2}\frac{C}{1-\lambda^2}} - \lambda\frac{3}{2}\frac{C}{1-\lambda^2}}, \\ b_3 &= \sqrt{\lambda\frac{3}{2}\frac{C}{1-\lambda^2} - (1-\lambda)\sqrt{\frac{3}{2}\frac{C}{1-\lambda^2}}} \end{aligned}\right\} \quad \ldots\ldots (23)$$

Following which the formulae (19^{a-c}) and (20^{a-c}) become :

$$\frac{y}{\nu P} = \frac{3}{4}e^{-b_1\xi}\frac{b_2\cos b_2\xi + b_1 \sin b_2\xi}{a^2 b_1 b_2} \quad \text{for } x \geq 0 \text{ and } \alpha > \delta, \quad \ldots\ldots (24^a)$$

$$\frac{y}{\nu P} = \frac{3}{4}e^{-a\xi}\frac{2 + a\xi\sqrt{2}}{a^3\sqrt{2}} \quad \text{for } x \geq 0 \text{ and } \alpha = \delta, \quad \ldots\ldots (24^b)$$

$$\frac{y}{\nu P} = \frac{3}{4}e^{b_1\xi}\frac{b_3 \operatorname{ch} b_3\xi + b_1 \operatorname{sh} b_3\xi}{a^2 b_1 b_3} \quad \text{for } x \geq 0 \text{ and } \alpha < \delta, \quad \ldots\ldots (24^c)$$

$$\frac{M}{Pl} = -\frac{1}{4} e^{-b_1 \xi} \frac{b_2 \cos b_2 \xi - b_1 \sin b_2 \xi}{b_1 b_2} \quad \text{for } x \geq 0 \text{ and } \alpha > \delta, \quad \ldots \quad (25^a)$$

$$\frac{M}{Pl} = -\frac{1}{4} e^{-b_1 \xi} \frac{1 - a\xi\sqrt{2}}{a\sqrt{2}} \quad \text{for } x \geq 0 \text{ and } \alpha = \delta, \quad \ldots \quad (25^b)$$

$$\frac{M}{Pl} = -\frac{1}{4} e^{-b_1 \xi} \frac{b_3 \operatorname{ch} b_3 \xi - b_1 \operatorname{sh} b_3 \xi}{b_1 b_3} \quad \text{for } x \geq 0 \text{ and } \alpha < \delta. \quad \ldots \quad (25^c)$$

9. Comparison of the results obtained from Chapters 5-8.

In the foregoing calculations, we have found the formulae by which the deflections of the bar and the bending moments can be calculated for disconnected bearings and

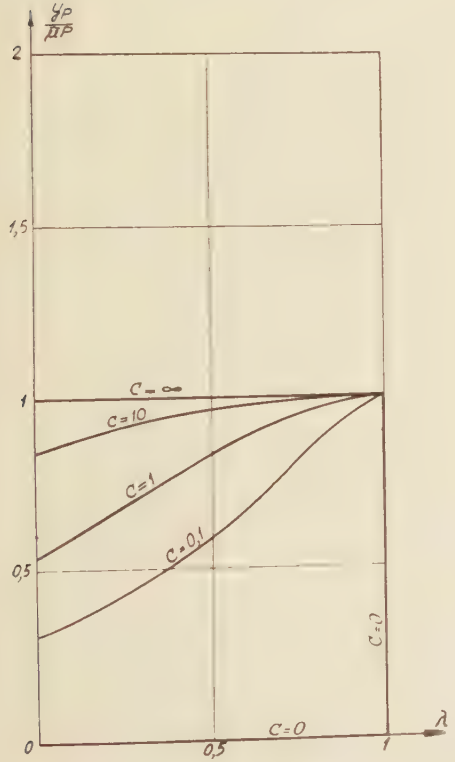


Fig 6*a*. — Deflection at the point of application of the force as a function of λ for $\xi_p = 0$.

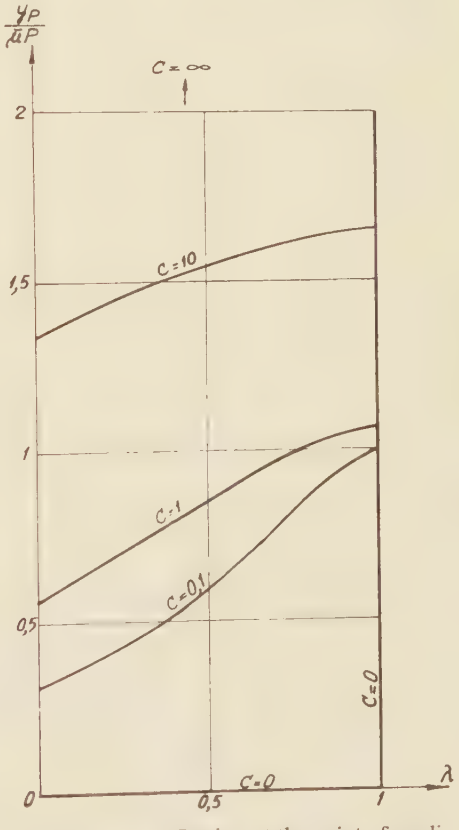


Fig. 6*b*. — Deflection at the point of application of the force as a function of λ for $\xi_p = 1/2$.

also for continuous bearings. To examine the respects in which these quantities differ from the corresponding quantities calculated for a continuous bed whose resilience constants are determined in accordance with Chapter 8, we shall calculate these quantities numerically for several values of C and λ.

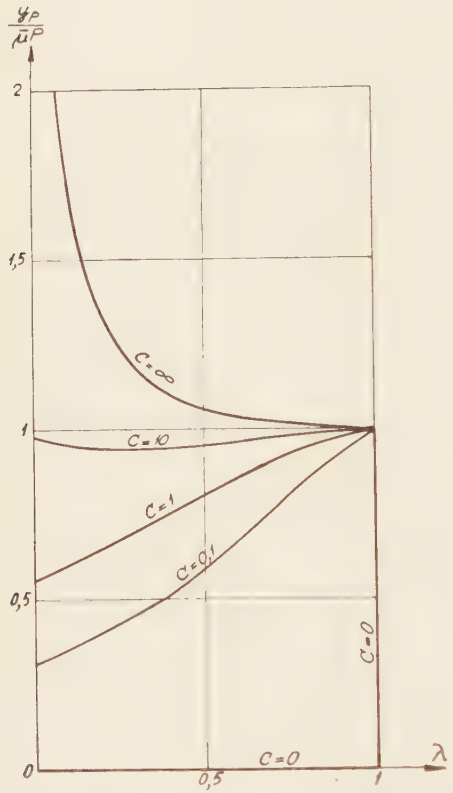


Fig. 6c. — Deflection at the point of application of the force as a function of λ for the continuous case.

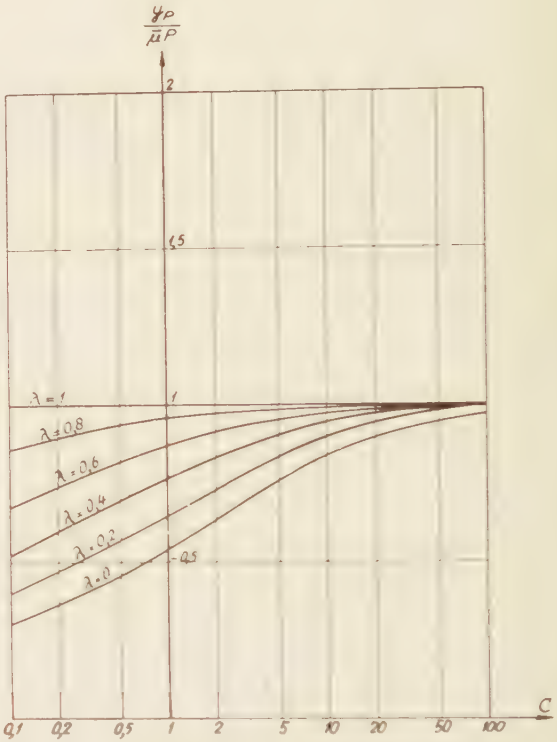


Fig. 7a. — Deflection at the point of application of the force as a function of C for $\xi_p = 0$.

In the first place, we shall determine the deflection and bending moment at the point of application of the force. For this purpose, it is convenient to introduce some further abbreviations. Taking :

$$\psi = \sqrt{8\,\frac{1-\lambda^2}{C} + (1+\lambda)^2}, \quad \chi = \sqrt{4(1+\lambda+\lambda^2) + 2\psi\sqrt{3(1-\lambda)\psi}} \quad . \quad . \quad . \quad . \quad (1)$$

we have in accordance with (**4**, 18^a) and (**4**, 18^b), as can be readily verified :

$$\operatorname{ch}\delta = \frac{\psi + (1-\lambda)\sqrt{3}}{2\sqrt{2}\sqrt{\frac{1-\lambda^2}{C} - \lambda}}, \quad \cos\varphi = \frac{\psi - (1-\lambda)\sqrt{3}}{2\sqrt{2}\sqrt{\frac{1-\lambda^2}{C} - \lambda}}, \quad \ldots\ldots\ldots\ldots (2^a)$$

and then :

$$\operatorname{sh}\delta = \frac{\chi}{2\sqrt{2}\sqrt{\frac{1-\lambda^2}{C} - \lambda}}, \quad \frac{e^{-\delta}}{\operatorname{sh}\delta} = \frac{\psi + (1-\lambda)\sqrt{3}}{\chi} - 1. \quad \ldots\ldots\ldots\ldots (2^b)$$

The formulae (**7**, 6) and (**7**, 7) for the deflection and the bending moment at the point of application of the force can then be established in forms valid also for regions II-IV of the plane (C, λ).

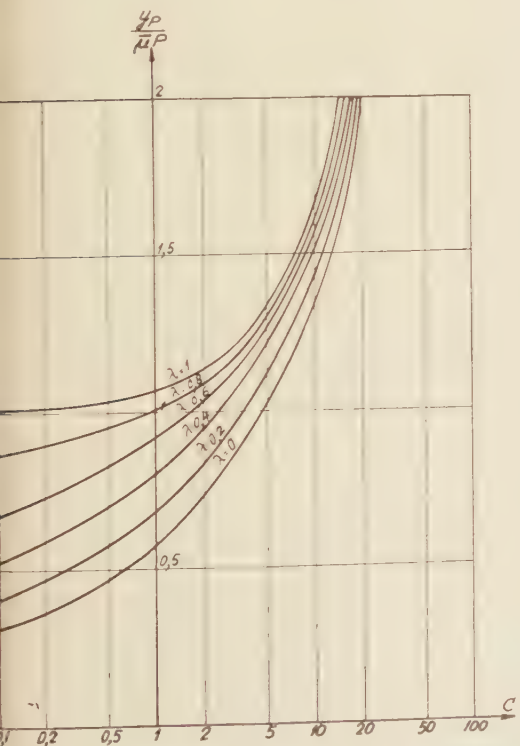


Fig. 7*b*. — Deflection at the point of application of the force as a function of C for $\xi_p = 1/2$.

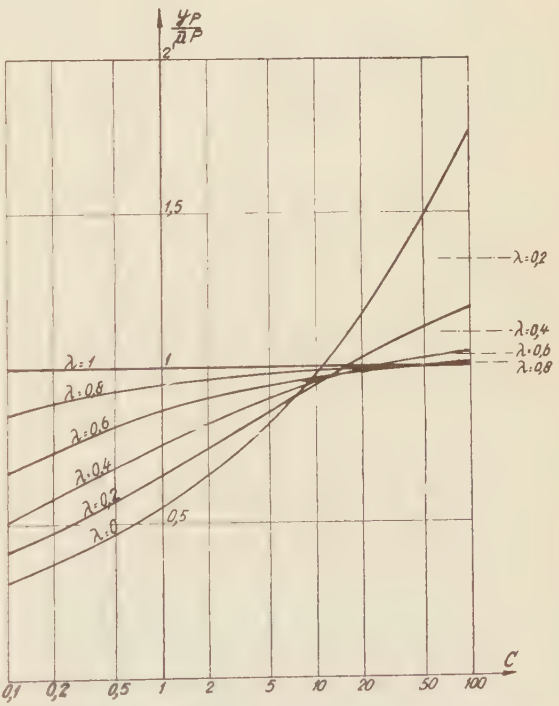


Fig. 7*c*. — Deflection at the point of application of the force as a function of λ for the continuous case.

$$\left.\begin{aligned} \frac{y_p}{\nu P} &= \frac{1+\lambda}{C\psi}\frac{\psi\sqrt{3}+1-\lambda}{\chi}, \\ \frac{M_p}{Pl} &= -\frac{2}{\psi\chi}\frac{1-\lambda^2}{C}, \end{aligned}\right\} \quad \text{for } \xi_p = 0, \qquad (3)$$

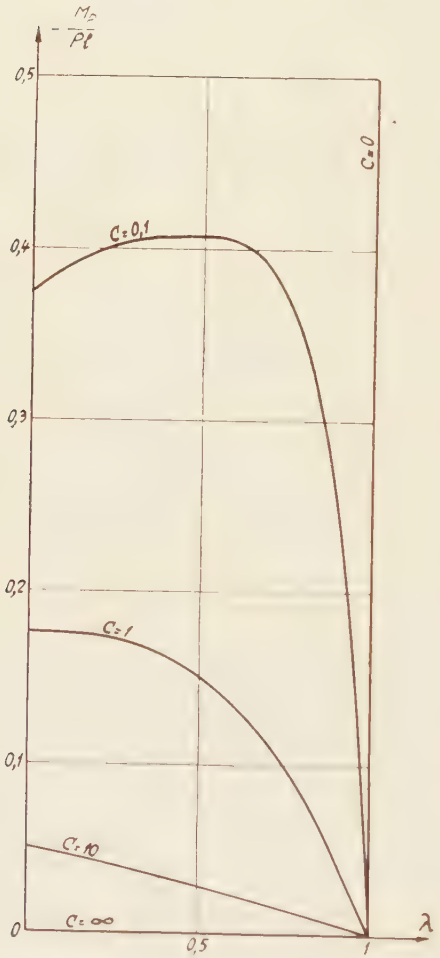


Fig. 8*a*. — Bending moment at the point of application of the force as a function of λ for $\xi_p = 0$.

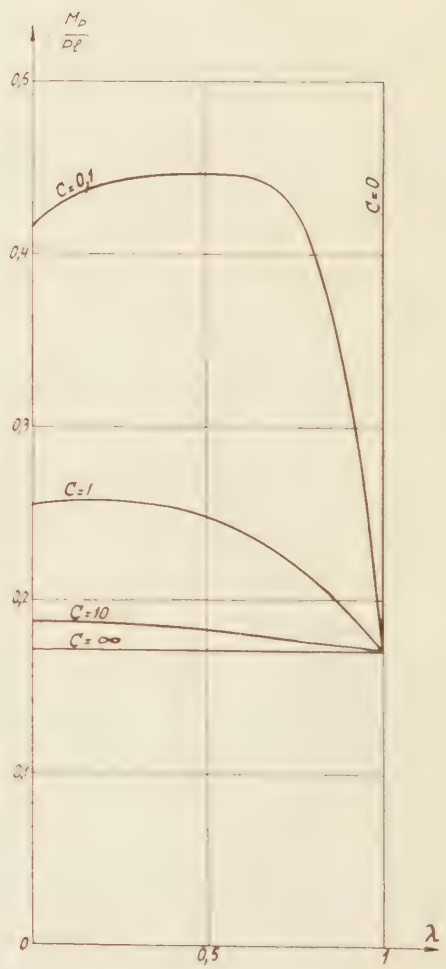


Fig. 8*b*. — Bending moment at the point of application of the force as a function of λ for $\xi_p = 1/2$.

$$\left.\begin{aligned}
\frac{y_p}{\nu P} &= \frac{1}{768}\,\frac{\psi+(1-\lambda)\sqrt{3}-\chi}{\left(\frac{1-\lambda^2}{C}-\lambda\right)\chi}\left\{384\,\frac{1-\lambda^2}{C}+21-6\lambda+21\lambda^2\right.\\
&\left.+3\sqrt{3}\,(1-\lambda)(5\psi+\chi)+12\psi\chi+\frac{96\sqrt{3}\,(1+\lambda)(\psi+\chi)}{C}\right\},\\
\frac{M_p}{Pl} &= -\frac{1}{64}\,\frac{\psi+(1-\lambda)\sqrt{3}-\chi}{\left(\frac{1-\lambda^2}{C}-\lambda\right)\chi}\left\{16\,\frac{1-\lambda^2}{C}+5+2\lambda+5\lambda^2\right.\\
&\left.+\sqrt{3}\,(1-\lambda)(3\psi+\chi)+2\psi\chi\right\},
\end{aligned}\right\}\quad \text{for } \xi_p=\frac{1}{2}. \quad (4)$$

Taking, in these formulae $\lambda = 1$, they have the more simple forms

$$\psi = 2, \quad \chi = 2\sqrt{3}, \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

$$\left.\begin{aligned}
\frac{y_p}{\nu P} &= \frac{1}{C}, \quad \frac{M_p}{Pl} = 0 && \text{for } \xi_p = 0, \quad \ldots\ldots\ldots\ldots \quad (6^a)\\
\frac{y_p}{\nu P} &= \frac{3\sqrt{3}-1}{64}+\frac{1}{C}, \quad \frac{M_p}{Pl} = -\frac{\sqrt{3}+1}{16} && \text{for } \xi_p = \frac{1}{2}, \quad \ldots\ldots\ldots\ldots \quad (6^b)
\end{aligned}\right\}$$

whilst for $C = \infty$ they are converted to :

$$\psi = 1+\lambda, \quad \chi = (\sqrt{3}+1)+(\sqrt{3}-1)\lambda, \quad \ldots\ldots\ldots \quad (7)$$

$$\left.\begin{aligned}
\frac{y_p}{\nu P} &= 0, \quad \frac{M_p}{Pl} = 0 && \text{for } \xi_p = 0, \quad \ldots\ldots\ldots\ldots \quad (8^a)\\
\frac{y_p}{\nu P} &= \frac{3\sqrt{3}-1}{64}, \quad \frac{M_p}{Pl} = -\frac{\sqrt{3}+1}{16} && \text{for } \xi_p = \frac{1}{2}. \quad \ldots\ldots\ldots\ldots \quad (8^b)
\end{aligned}\right\}$$

By virtue of the physical conditions, we can easily set out a method of control for these formulae. In effect, for $C = \infty$, the depression of each bearing is equal to zero, and for $\lambda = 1$ it is equal to $\bar{\mu}P \cdot \frac{\nu P}{C}$. The deformation of the bar in relation to the

bearings is thus the same in the two cases and the absolute values of the depressions in the second case exceed the values of those in the first case by a quantity $\frac{\nu P}{C}$.

The corresponding formulae for the case of the bar on a continuous bed are in accordance with (**8**, 23), (**8**, 24*a—c*) and (**8**, 25*a—c*).

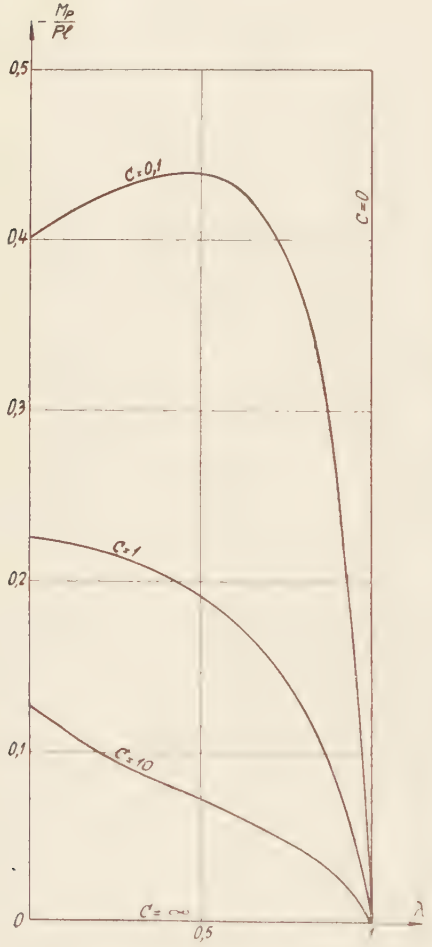
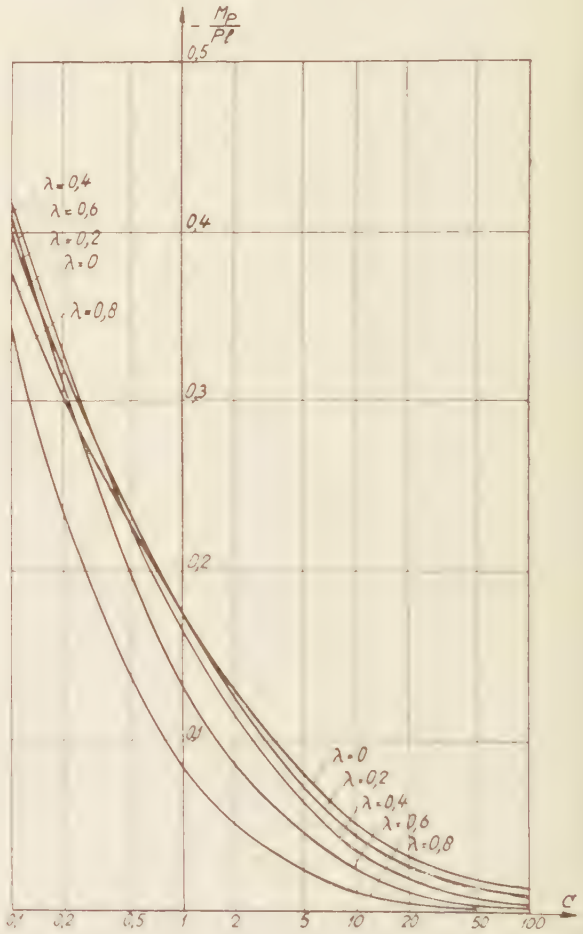


Fig. 8*c*. — Bending moment at the point of application of the force as a function of λ for the continuous case.



N. B. — Add "λ = 1" to the axis C.

Fig. 9*a*. — Bending moment at the point of application of the force as a function of C for $\xi_p = 0$.

$$\frac{y}{\nu P} = \frac{3}{4\sqrt{\frac{3}{2}C\,\frac{1-\lambda}{1+\lambda}\left[(1-\lambda)\sqrt{\frac{\frac{3}{2}C}{1-\lambda^2}}+\lambda\sqrt{\frac{\frac{3}{2}C}{1-\lambda^2}}\right]}}, \qquad (9a)$$

$$\frac{M_p}{Pl} = \frac{1}{4\sqrt{(1-\lambda)\sqrt{\frac{\frac{3}{2}C}{1-\lambda^2}}+\lambda\sqrt{\frac{\frac{3}{2}C}{1-\lambda^2}}}}. \qquad (9b)$$

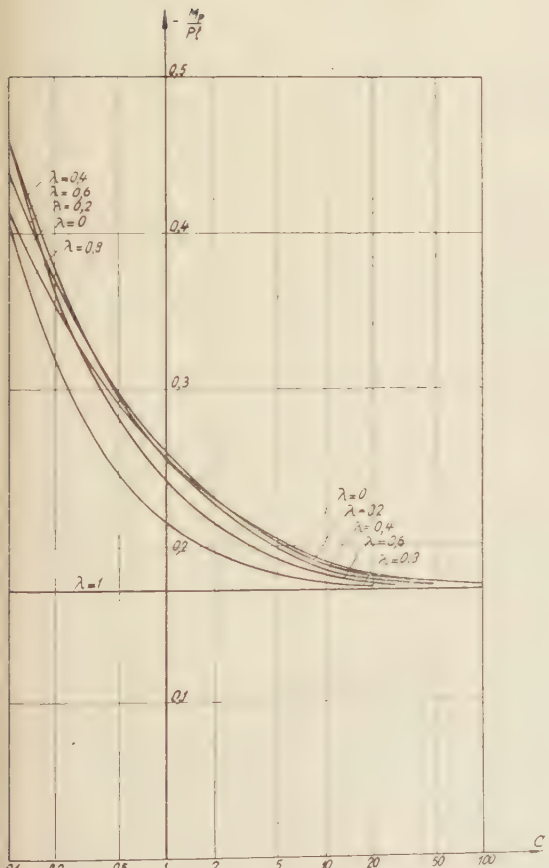


Fig. 9*b*. — Bending moment at the point of application of the force as a function of C for $\xi_p = 1/2$.

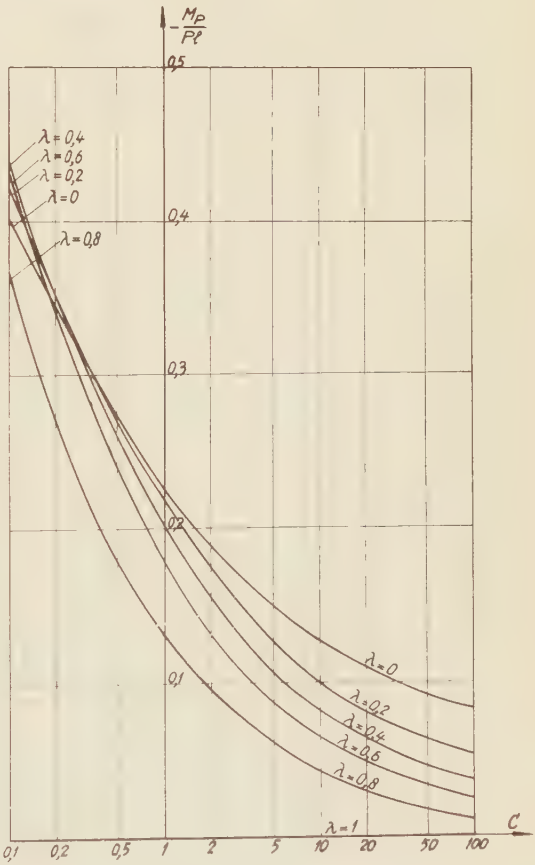


Fig. 9*c*. — Bending moment at the point of application of the force as a function of λ for the continuous case.

TABLE III*a*.

Some values of the factor ψ.

C \ λ	0	0.2	0.4	0.6	0.8	1
0	∞	∞	∞	∞	∞	$2\sqrt{4\frac{1-\lambda}{C}+1}$
0.1	9	8.8453	8.3162	7.3321	5.6604	2
1	3	3.0199	2.9462	2.7713	2.4739	2
10	1.3416	1.4859	1.6223	1.7527	1.8783	2
∞	1	1.2	1.4	1.6	1.8	2

TABLE III*b*.

Some values of the factor χ.

C \ λ	0	0.2	0.4	0.6	0.8	1
0	∞	∞	∞	∞	∞	$2\sqrt{3+\sqrt{3}(1-\lambda)\sqrt{4\frac{1-\lambda}{C}+1}}$
0.1	5.9310	5.4289	4.8503	4.2426	3.6989	3.4641
1	3.7937	3.6509	3.5162	3.4176	3.3873	3.4641
10	2.9407	3.0130	3.1003	3.2045	3.3259	3.4641
∞	2.7321	2.8785	3.0249	3.1713	3.3177	3.4641

For several values of C and λ we have calculated the factors ψ and χ and then the deflections y_p and the moments M_p. This shews the results in Tables III*a*-IV*b* and figures 6*a*-9*c*. We have here resolved the quantities without dimensions $\frac{y_p}{\mu P}$ (which equal from (4, 9) $C\frac{y_p}{\nu P}$) but not those without dimensions $\frac{y_p}{\nu P}$.

TABLE IV*a*.

Some values of the deflection at the point of application of the force $\frac{y_p}{\mu P}$.

	C \ λ	0	0.2	0.4	0.6	0.8	1
$\xi_p = 0$	0	0	0	0	0	0	undifined
	0.1	0.3108	0.4028	0.5208	0.6738	0.8601	1
	1	0.5444	0.6564	0.7707	0.8785	0.9634	1
	10	0.8425	0.9043	0.9491	0.9788	0.9950	1
	∞	1	1	1	1	1	1
$\xi_p = \frac{1}{2}$	0	0	0	0	0	0	undifined
	0.1	0.3119	0.4040	0.5220	0.6752	0.8620	1.0066
	1	0.5697	0.6837	0.8017	0.9154	1.0134	1.0656
	10	1.3434	1.4359	1.5130	1.5747	1.6217	1.6557
	∞	0.7010 + 0.0656 C	0.7831 + 0.0637 C	0.9140 + 0.0616 C	0.9108 + 0.0630 C	0.9596 + 0.0639 C	1 + 0.0656 C
Continuous method	0	0	0	0	0	0	undifined
	0.1	0.3112	0.4023	0.5189	0.6700	0.8543	1
	1	0.5533	0.6547	0.7597	0.8620	0.9498	1
	10	0.9840	0.9459	0.9508	0.9683	0.9873	1
	∞	∞	1.3417	1.1067	1.0328	1.0063	1

TABLE IV*b*.

Some values of the bending moment at the point of application of the force $-\frac{M_p}{Pl}$.

	C \ λ	0	0.2	0.4	0.6	0.8	1
$\xi_p = 0$	0	∞	∞	∞	∞	∞	undifined
	0.1	0.3747	0.3998	0.4165	0.4115	0.3439	0
	1	0.1757	0.1741	0.1622	0.1351	0.0859	0
	10	0.0507	0.0429	0.0334	0.0228	0.0115	0
	∞	0	0	0	0	0	0
$\xi_p = \frac{1}{2}$	0	∞	∞	∞	∞	∞	undifined
	0.1	0.4159	0.4404	0.4583	0.4582	0.4054	0.1708
	1	0.2548	0.2576	0.2540	0.2408	0.2144	0.1708
	10	0.1877	0.1864	0.1839	0.1803	0.1759	0.1708
	∞	0.1708	0.1708	0.1708	0.1708	0.1708	0.1708
Continuous method	0	∞	∞	∞	∞	∞	undifined
	0.1	0.4017	0.4241	0.4385	0.4324	0.3676	0
	1	0.2259	0.2182	0.2030	0.1760	0.1292	0
	10	0.1270	0.0997	0.0804	0.0625	0.0425	0
	∞	0	0	0	0	0	0

In particular :

$$\left.\begin{aligned} &\frac{y_p}{\bar{\mu}P} = 1 && \text{for } \xi_p = 0,\\ &\frac{y_p}{\bar{\mu}P} = 1 + \frac{3\sqrt{3}-1}{64}\,C && \text{for } \xi_p = \frac{1}{2},\\ &\frac{y_p}{\bar{\mu}P} = 1 && \text{for the continuous case,} \end{aligned}\right\} \text{and } \lambda = 1, \quad \ldots\ldots \quad (10)$$

$$\left.\begin{aligned} &\psi = \infty,\quad \psi\sqrt{C} = 2\sqrt{2(1-\lambda^2)},\quad \psi C = 0,\\ &\chi = \infty,\quad \chi\sqrt{C} = 0,\quad \frac{\psi}{\chi} = \infty,\quad \frac{\psi}{\chi}\sqrt{C} = 0,\\ &\frac{y_p}{\bar{\mu}P} = 0 \qquad \text{for } \xi_p = 0,\\ &\frac{y_p}{\bar{\mu}P} = 0 \qquad \text{for } \xi_p = \frac{1}{2},\\ &\frac{y_p}{\bar{\mu}P} = 0 \qquad \text{for the continuous case,} \end{aligned}\right\} \text{and } C = 0, \quad \ldots\ldots \quad (11)$$

$$\left.\begin{aligned} &\frac{y_p}{\bar{\mu}P} = 1 && \text{for } \xi_p = 0 \text{ and } C = \infty,\\ &\lim_{C\to\infty} \frac{y_p}{\bar{\mu}P} = \\ &\frac{16\,(1+\lambda)\,(7+2\sqrt{3}-\lambda)+\{7+5\sqrt{3}+(-4+3\sqrt{3})\,\lambda+3\lambda^2\}C}{64\,(2+\sqrt{3}+\lambda)} && \text{for } \xi_p = \frac{1}{2},\\ &\frac{y_p}{\bar{\mu}P} = \frac{1+\lambda}{2\sqrt{\lambda}} && \text{for the continuous case and } C = \infty. \end{aligned}\right\} \quad (12)$$

We have next calculated the deflections and bending moments at various places of the bar for the values $C = 2\sqrt{15}$, $\lambda = 4 - \sqrt{15}$, which relate to point A of figure 3. We have here used formulae (**8**, 12), (**8**, 13), (**8**, 24) and (**8**, 25), in which, according to (**8**, 23) :

$$a = \sqrt{3}, \quad b_1 = 3\sqrt{\frac{1}{2}(-3+\sqrt{15})}, \quad b_2 = \sqrt{\frac{3}{2}(13-3\sqrt{15})}.$$

TABLE V*a*.

Values of the deflections for the case $C = 2\sqrt{15}$, $\lambda = 4 - \sqrt{15}$.

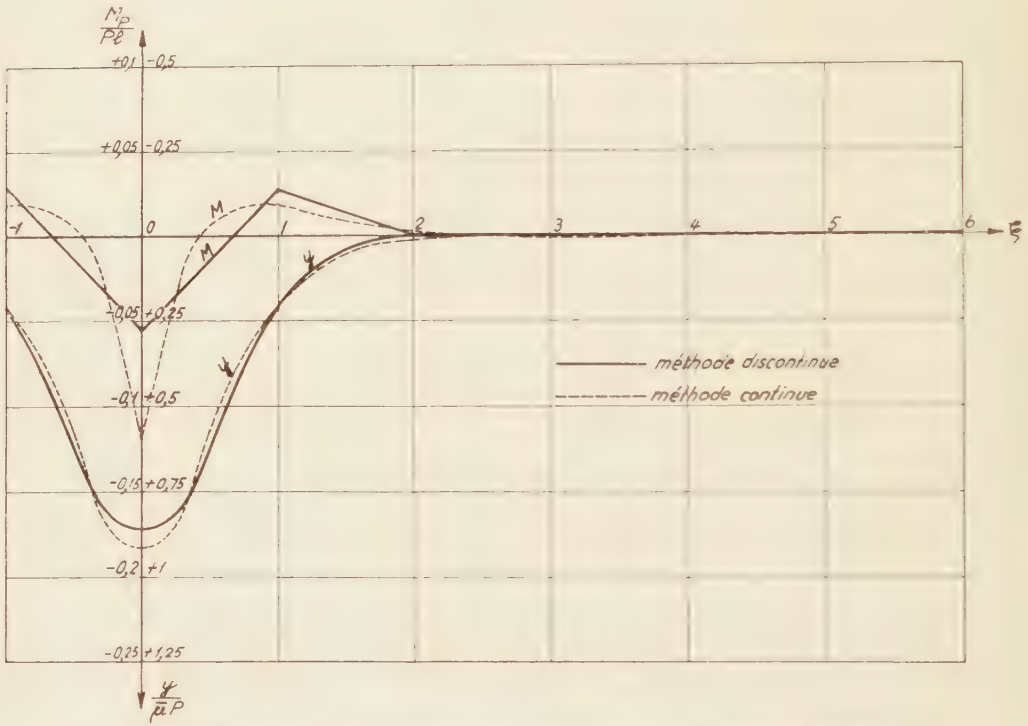
$\xi_p = 0$		$\xi_p = \frac{1}{2}$		Case continuous	
n	$\frac{y_n}{\mu P}$	n	$\frac{y_n}{\mu P}$	ξ	$\frac{y(\xi)}{\mu P}$
0	0.8607	1	0.6186	0	+ 0.9129
1	0.2152	2	0.0027	0.5	+ 0.5734
2	0	3	0	1	+ 0.2154
3	0	4	0	1.5	+ 0.0533
4	0	.	.	2	+ 0.0044
.	.	.	.	2.5	— 0.0039
.	.	.	.	3	— 0.0027
.	.			3.5	— 0 0011
		$\frac{y_p}{\mu P}$	1.2439	4	— 0.0003
				4.5	— 0.0000
				5	— 0.0000
				5.5	— 0.0000

TABLE V*b*.

Values of the bending moments for the case $C = 2\sqrt{15}$, $\lambda = 4 - \sqrt{15}$.

$\xi_p = 0$		$\xi_p = \frac{1}{2}$		Case continuous	
n	$\frac{M}{Pl}$	n	$\frac{M}{Pl}$	ξ	$\frac{M(\xi)}{Pl}$
0	— 0.0556	1	+ 0.0590	0	— 0.1179
1	+ 0 0278	2	+ 0.0035	0.5	+ 0.0072
2	0	3	0	1	+ 0.0182
3	0	4	0	1.5	+ 0.0095
4	0	.	.	2	+ 0.0032
.	.	.	.	2.5	+ 0.0007
.	.	.	.	3	— 0.0000
				3.5	— 0.0001
				4	— 0.0000
		$\frac{M_p}{Pl}$	— 0.1910	4.5	— 0.0000
				5	— 0.0000
				5.5	— 0.0000

Here we have also substituted $\frac{y}{\mu P}$ for $\frac{y}{\nu P}$. The results are found in Tables V^a, V^b and figures 10^a and 10^b.



Méthode discontinue = discontinuous method. — méthode continue = continuous method.

Fig. 10*a*. — Deflection and bending moment for $C = 2\sqrt{15}$, $\lambda = 4 - \sqrt{15}$, $\xi_p = 0$.

Finally, we have determined the deflections and bending moments at several points of the bar for some other values of C and λ. We have selected for this the cases $\lambda = 0$ and $\lambda = 1$ and the values $C = 0$, 0.1, 1, 10 and ∞. In the first case, we have used the formulae (7, 4), (7, 5), (8, 23), (8, 24^a) and (8, 25^a), except for $C = \infty$, for which value $\varepsilon_1 \ldots \varepsilon_4$ (4, 7) are actual, so that it is necessary to apply the formulae (5, 12) and (5, 13). From (4, 2) and (4, 5-7) we obtain :

$$\left.\begin{array}{c} \theta_1 = +\infty, \qquad \theta_2 = -6 \\ \varepsilon_1 = 0,\ \varepsilon_2 = -2 - \sqrt{3},\ \varepsilon_3 = +\infty,\ \varepsilon_4 = -2 + \sqrt{3} \end{array}\right\} \text{ for } \lambda = 0 \text{ and } C = \infty. \quad (13)$$

In order to use formulae (5, 12) and (5, 13) (in which it is then necessary to substitute

ε_4 for ε_2). We must also calculate the limit-value of $C\ (1 + 4\ \varepsilon_4 + \varepsilon_4^2)$ for $C \to \infty$. By series development, we find for this quantity :

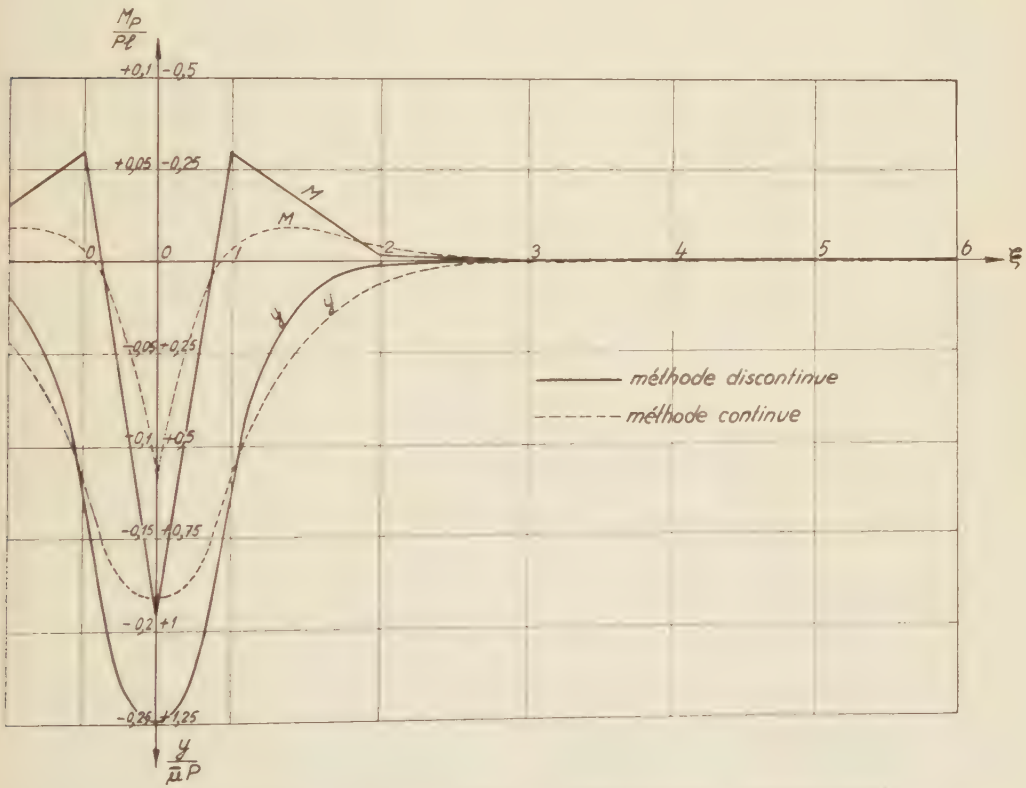
$$\lim_{C \to \infty} C(1 + 4\,\varepsilon_4 + \varepsilon_4^2) = C(\theta_2 + 6)\varepsilon_4$$

$$= \frac{1}{2}\ \varepsilon_4 C\left(12 - C + C\sqrt{1 - \frac{24}{C}}\right)$$

$$= \frac{1}{2}\ \varepsilon_4 C\left(12 - C + C - 12 - \frac{1}{8}\ C.\ \frac{24^2}{C^2} \ldots\right) \quad \text{for } \lambda = 0,$$

so that we get finally :

$$\lim_{C \to \infty} C(1 + 4\,\varepsilon_4 + \varepsilon_4^2) = -\,36\,\varepsilon_4 = 36\,(2 - \sqrt{3}) \qquad \text{for } \lambda = 0. \tag{14}$$



Méthode discontinue = discontinuous method. — méthode continue = continuous method.

Fig. 10*b*. — Deflection and bending moment for $C = 2\sqrt{15}$, $\lambda = 4 - \sqrt{15}$, $\xi_p = 1/2$.

TABLE VI*a*.

Some values of the deflection of the bar for $\lambda = 0$.

ξ_p	C	$\frac{y_p}{\mu P}$	$\frac{y_o}{\bar{\mu} P}$	$\frac{y_1}{\bar{\mu} P}$	$\frac{y_2}{\bar{\mu} P}$	$\frac{y_3}{\bar{\mu} P}$	$\frac{y_4}{\bar{\mu} P}$	$\frac{y_5}{\bar{\mu} P}$
0	0	0	0	0	0	0	0	0
	0.1	+ 0.3108	+ 0.3108	+ 0.2327	+ 0.1200	+ 0.0320	— 0.0048	— 0.0134
	1	+ 0.5443	+ 0.5443	+ 0.2451	+ 0.0130	— 0.0231	— 0.0083	— 0.0002
	10	+ 0.8424	+ 0.8424	+ 0.1094	— 0.0328	+ 0.0018	+ 0.0005	— 0.0001
	∞	+ 1	+ 1	0	0	0	0	0
$\frac{1}{2}$	0	0	0	0	0	0	0	0
	0.1	+ 0.3119	+ 0.2869	+ 0.2869	+ 0.1712	+ 0.0667	+ 0.0089	— 0.0115
	1	+ 0.5697	+ 0.4412	+ 0.4412	+ 0.0965	— 0.0199	— 0.0165	— 0.0027
	10	+ 1.3434	+ 0.5607	+ 0.5607	— 0.0576	— 0.0049	+ 0.0019	— 0.0001
	∞	+ 0.7010 + + 0.0656 C	+ 0.6005	+ 0.6005	— 0.1274	+ 0.0341	— 0.0091	— 0.0024

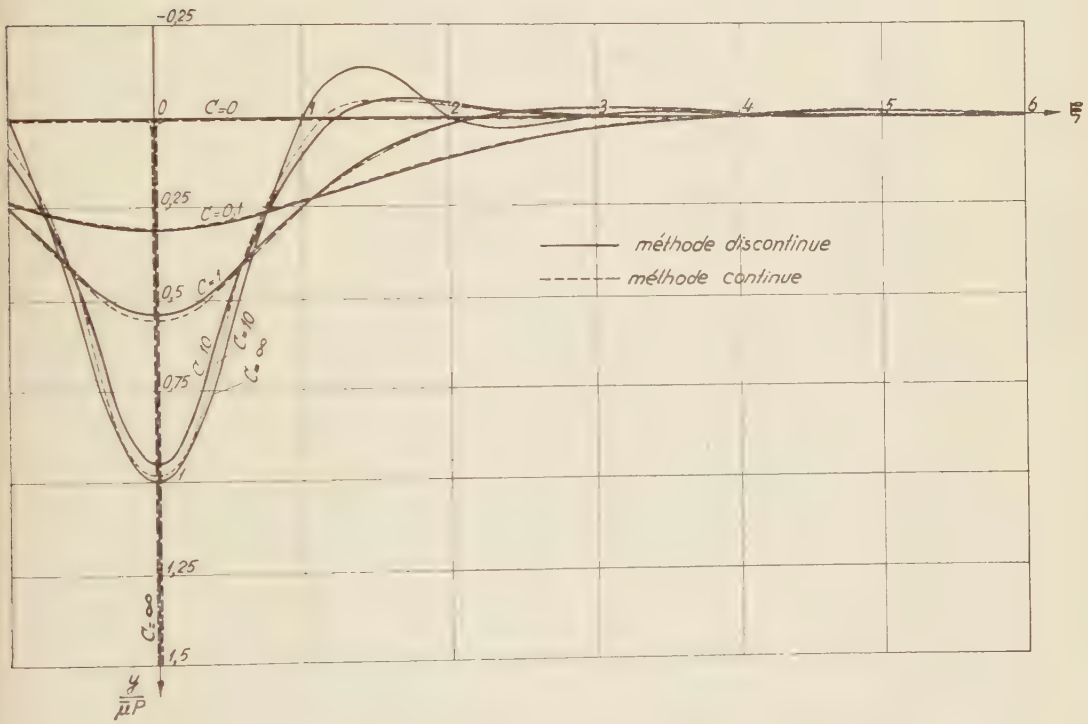
TABLE VI*b*.

Some values of the bending moment of the bar for $\lambda = 0$.

ξ_p	C	$\frac{M_p}{Pl}$	$\frac{M_o}{Pl}$	$\frac{M_1}{Pl}$	$\frac{M_2}{Pl}$	$\frac{M_3}{Pl}$	$\frac{M_4}{Pl}$	$\frac{M_5}{Pl}$
0	0	— ∞	— ∞	— ∞	— ∞	— ∞	— ∞	— ∞
	0.1	— 0.3747	— 0.3747	— 0.0300	+ 0.0817	— 0.0569	— 0.0462	— 0.0173
	1	— 0.1757	— 0.1757	+ 0.0521	+ 0.0348	— 0.0046	— 0.0025	— 0.0012
	10	— 0.0507	— 0.0507	+ 0.0281	— 0.0025	— 0.0003	— 0.0001	— 0.0000
	∞	0	0	0	0	0	0	0
$\frac{1}{2}$	0	— ∞	— ∞	— ∞	— ∞	— ∞	— ∞	— ∞
	0.1	— 0.4159	— 0.1657	— 0.1657	+ 0.0472	+ 0.0889	— 0.0640	— 0.0302
	1	— 0.2548	— 0.0048	— 0.0048	+ 0.0541	+ 0.0166	— 0.0010	— 0.0021
	10	— 0.1877	+ 0.0623	+ 0.0623	+ 0.0016	— 0.0016	— 0.0002	— 0.0000
	∞	— 0.1708	+ 0.0792	+ 0.0792	— 0.0212	+ 0.0057	— 0.0015	— 0.0004

We thus deduce from (**5**, 12) and (**5**, 13)

$$\left.\begin{aligned} &\frac{y_0}{\bar{\mu}P} = 1, \quad \frac{y_n}{\bar{\mu}P} = 0\ (n \geq 1), \quad \frac{M_n}{Pl} = 0\ (n \geq 0) \text{ for } \xi_p = 0,\ \lambda = 0 \text{ and } C = \infty, \quad (15^a)\\ &\frac{y_1}{\bar{\mu}P} = \frac{10 - 3\sqrt{3}}{8}, \quad \frac{y_n}{\bar{\mu}P} = \frac{9 - 3\sqrt{3}}{8}\varepsilon_4{}^{n-1}(n \geq 2), \quad \frac{M_n}{Pl} = \frac{3 - \sqrt{3}}{16}\varepsilon_4{}^{n-1}(n \geq 1)\\ &\text{for } \xi_p = \frac{1}{2},\ \lambda = 0 \text{ and } C = \infty. \quad (15^b) \end{aligned}\right\}$$



Méthode discontinue = discontinuous method. — méthode continue = continuous method.

Fig. 11*a*. — Deflection for the case $\lambda = 0$, $\xi_p = 0$.

In Tables VI^a-VI^c are found the calculated values of the deflections and bending moments, which are shewn graphically in figures 11^a-12^b as functions of the co-ordinates ξ. To trace the curve for $C = \infty$ in the case $\xi_p = 0$ in figure 11^a, we have calculated y not only for $\xi = 0, 1, 2$, etc., but also for several intermediate values of ξ. This can be done in the following manner.

TABLE VIc.

Some values of the deflection and bending moment as a function of the co-ordinate ξ for the continuous case and $\lambda = 0$.

C	0		0.1		1		10		∞	
ξ	$\frac{y}{\mu P}$	$\frac{M}{Pl}$	$\frac{y}{\mu P}$	$\frac{M}{Pl}$	$\frac{y}{\mu P}$	$\frac{M}{Pl}$	$\frac{y}{\mu P}$	$\frac{M}{Pl}$	$\frac{y}{\mu P}$	$\frac{M}{Pl}$
0	0	$-\infty$	+ 0.3112	— 0.4017	+ 0.5534	— 0.2259	+ 0.9840	— 0.1270	$+\infty$	0
0.5	0	$-\infty$	+ 0.2868	— 0.1903	+ 0.4379	— 0.0423	+ 0.5114	+ 0.0132	0	0
1	0	$-\infty$	+ 0.2330	— 0.0495	+ 0.2455	+ 0.0333	+ 0.0736	+ 0.0232	0	0
1.5	0	$-\infty$	+ 0.1706	+ 0.0329	+ 0.0954	+ 0.0466	— 0.0408	+ 0.0078	0	0
2	0	$-\infty$	+ 0.1136	+ 0.0725	+ 0.0122	+ 0.0346	— 0.0272	— 0.0000	0	0
2.5	0	$-\infty$	+ 0.0666	+ 0.0835	— 0.0196	+ 0.0184	— 0.0055	— 0.0011	0	0
3	0	$-\infty$	+ 0.0320	+ 0.0775	— 0.0232	+ 0.0066	+ 0.0015	— 0.0004	0	0
3.5	0	$-\infty$	+ 0.0088	+ 0.0633	— 0.0162	+ 0.0004	+ 0.0014	— 0.0000	0	0
4	0	$-\infty$	— 0.0048	+ 0.0467	— 0.0082	— 0.0018	+ 0.0004	+ 0.0000	0	0
4.5	0	$-\infty$	— 0.0115	+ 0.0312	— 0.0027	— 0.0019	— 0.0000	+ 0.0000	0	0
5	0	$-\infty$	— 0.0134	+ 0.0184	+ 0.0001	— 0.0012	— 0.0001	+ 0.0000	0	0

TABLE VII.

Some values of the deflection as a function of the co-ordinate ξ for the case $\lambda = 0$, $C = \infty$, $\xi_p = 0$.

$\xi = n$	$\frac{y}{\mu P}$				
	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
0	+ 1	0	0	0	0
0.2	+ 0.9217	— 0.1098	+ 0.0294	— 0.0079	+ 0.0021
0.4	+ 0.7252	— 0.1364	+ 0.0366	— 0.0098	+ 0.0026
0.6	+ 0.4678	— 0.1082	+ 0.0290	— 0.0078	+ 0.0021
0.8	+ 0.2069	— 0.0533	+ 0.0143	— 0.0038	+ 0.0010
1	0	0	0	0	0

The differential equation for y is according to (3, 1), (3, 6) and (3, 9)

$$\frac{d^2y}{dx^2} = \frac{1}{l^2} \quad \frac{d^2y}{d\xi^2} = \frac{M}{EI} = C\,\frac{M}{Pl} \cdot \frac{6\,\mu P}{l^2}$$

or

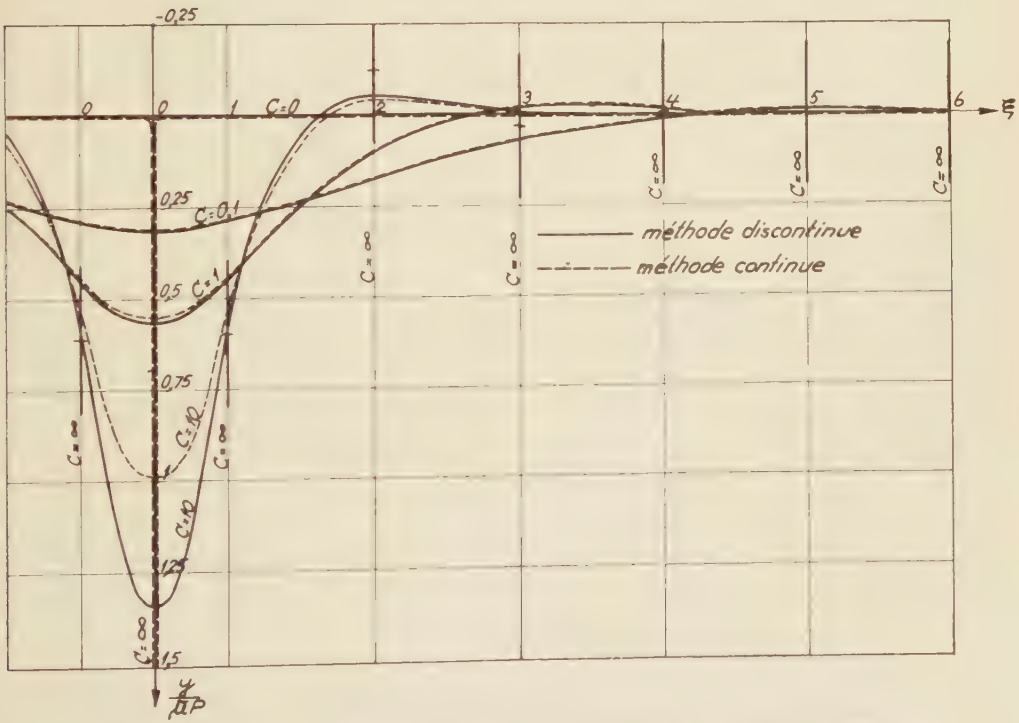
$$\frac{1}{\mu P}\,\frac{d^2y}{d\xi^2} \quad 6\,C\,\frac{M}{Pl} \cdot \ldots\ldots\ldots\ldots \quad (16)$$

However M varies linearly between two bearings, so that we can state :

$$M = M_n + (M_{n+1} - M_n)\,(\xi - n) \quad (n \leq \xi \leq n+1). \ldots\ldots \quad (17)$$

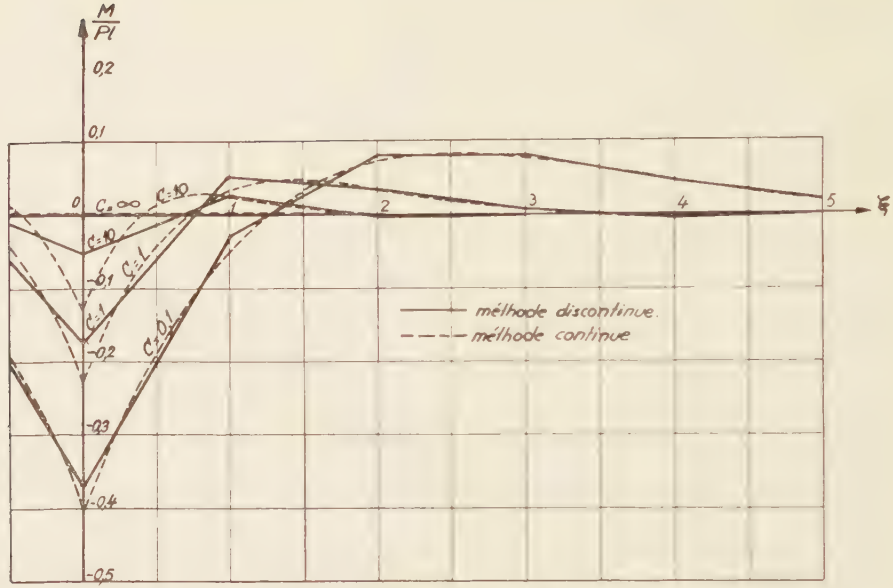
From (5, 12), we also get :

$$\frac{CM_o}{Pl} = -(\sqrt{3} - 1), \quad \frac{CM_n}{Pl} = -\sqrt{3}\,\varepsilon_4{}^n \quad (n \geq 1), \ldots\ldots \quad (18)$$



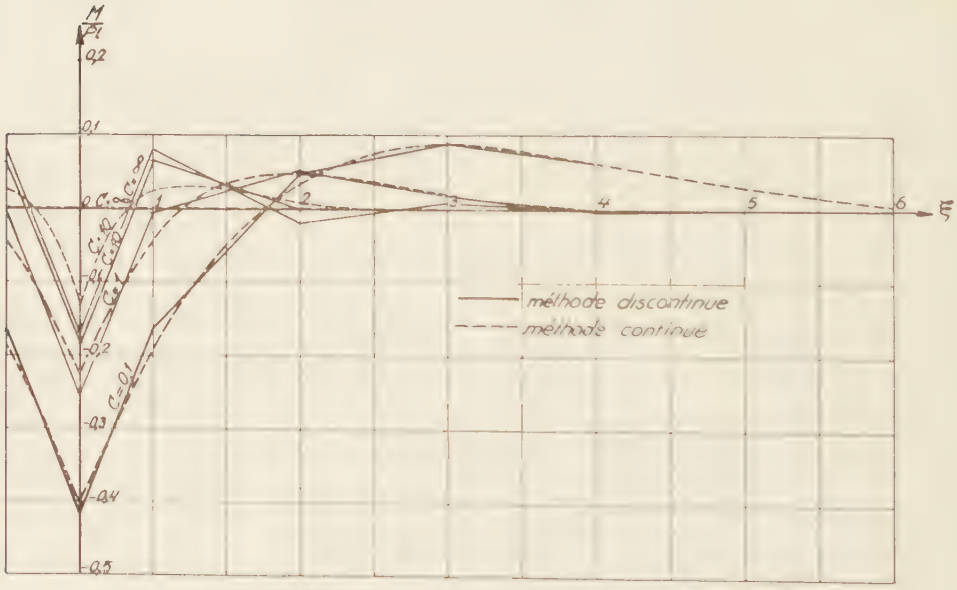
Méthode discontinue = discontinuous method, — méthode continue = continuous method.

Fig. 11*b*. — Deflection for the case $\lambda = 0$, $\xi_p = 1/2$.



Méthode discontinue = discontinuous method. — méthode continue = continuous method.

Fig. 12*a*. — Bending moment for the case $\lambda = 0$, $\xi_p = 0$.



Méthode discontinue = discontinuous method. — méthode continue — continuous method.

Fig. 12*b*. — Bending moment for the case $\lambda = 0$, $\xi_p = 1/2$.

and thus we find for the differential equation for y :

$$\left.\begin{aligned} \frac{1}{\bar{\mu}\mathrm{P}}\,\frac{d^2y}{d\xi^2} &= 6\{(3\sqrt{3}-4)\,\xi-(\sqrt{3}-1)\} && \text{for } 0\leq\xi\leq 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \frac{1}{\bar{\mu}\mathrm{P}}\,\frac{d^2y}{d\xi^2} &= 6\,\varepsilon_4{}^n\{(3\sqrt{3}-4)\,(\xi-n)-\sqrt{3}\} && \text{for } n\leq\xi\leq n+1\ (n\geq 1). \quad . \quad . \end{aligned}\right\} \quad (19)$$

The limit conditions are :

$$y=1 \text{ for } \xi=0, \quad y=0 \text{ for } \xi=n\ (n=1,2\ldots), \quad \frac{dy}{d\xi}=0 \text{ for } \xi=0; \quad . \quad . \quad (20)$$

after two integrations it is then deduced from (19) :

$$\left.\begin{aligned} \frac{y}{\bar{\mu}\,\mathrm{P}} &= (3\sqrt{3}-4)\,\xi^3-3\,(\sqrt{3}-1)\,\xi^2+1 && \text{for } 0\leq\xi\leq 1, \quad . \quad . \quad . \quad . \quad . \\ \frac{y}{\bar{\mu}\mathrm{P}} &= 3\,\varepsilon^n\{(\sqrt{3}-1)\,(\xi-n)^3-\sqrt{3}\,(\xi-n)^2+(\xi-n)\} \\ & && \text{for } n\leq\xi\leq n+1\ (n\geq 1). \end{aligned}\right\} \quad (21)$$

We have included in Table VII several values calculated from $\frac{y}{\bar{\mu}\mathrm{P}}$ (21).

For the case $\lambda=1$, the formulae for the deflection and bending moment are much simplified. In this case region IV is always valid (see fig. 3) so that it is necessary to make use of the formulae (**5,** 12) and (**5,** 13) in which :

$$\left.\begin{aligned} &\theta_1=0, \quad \theta_2=-6, \\ &\varepsilon_1=1, \quad \varepsilon_2=-2-\sqrt{3}, \quad \varepsilon_3=1, \quad \varepsilon_4=-2+\sqrt{3} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and then we must substitute ε_4 for ε_2. In this case, the expression $\frac{1-\varepsilon_1}{1-\lambda}$ is indeterminate. It can be calculated, with the aid of (**4,** 2) as follows :

$$\lim_{\lambda\to 1}\left(\frac{1-\varepsilon_1}{1-\lambda}\right)^2=\lim_{\lambda\to 1}\frac{\varepsilon_1\theta_1}{(1-\lambda)^2}.$$

In this formula, we can substitute 1 for ε_1 in the right-hand side, whilst from (**4,** 6), we get by series development :

$$\lim_{\lambda\to 1}\frac{\theta_1}{2}=\lim_{\lambda\to 1}\frac{24\,(1-\lambda)^2\left(\frac{1-\lambda^2}{C}-\lambda\right)}{-2\left(\frac{1-\lambda^2}{C}-\lambda\right)(1-8\,\lambda+\lambda^2)},$$

So that we find after calculation :

$$\lim_{\lambda \to 1} \frac{1 - \varepsilon_1}{1 - \lambda} = 1 \text{ for all values of C.}$$

Thus, from (**5**, 12) and (**5**, 13) there results :

$$\left.\begin{aligned} &\frac{y_n}{\mu P} \equiv 1, \quad \frac{M_n}{Pl} = 0 \text{ for } \xi_p = 0 \text{ and for all values of } n \text{ and C,} \\ &\frac{y_n}{\mu P} \equiv 1, \quad \frac{M_n}{Pl} = \frac{3 - \sqrt{3}}{16} \varepsilon_4^{n-1} \text{ for } \xi_p = \frac{1}{2} \text{ and for all values of } n \text{ and C,} \end{aligned}\right\} \quad (23)$$

In the continuous case, the region $\alpha < \lambda < \delta$ is valid for this value of λ: from (**9**, 23), we have here :

$$\left.\begin{aligned} &\lim_{\lambda \to 1} a^2 = \lim_{\lambda \to 1} \frac{1}{2} \sqrt{3\,C\,(1 - \lambda)}, \lim_{\lambda \to 1} b_1 = \lim_{\lambda \to 1} b_3 = b = \frac{1}{2} \sqrt{\frac{3\,C}{1 - \lambda}}, \\ &\lim_{\lambda \to 1} a^2 b_1 = \lim_{\lambda \to 1} a^2 b_3 = \lim_{\lambda \to 1} a^2 b = \frac{3}{4} C, \end{aligned}\right\} \quad (24)$$

and then from (**9**, 24^c) and (**9**, 25^c)

$$\left.\begin{aligned} &\lim_{\lambda \to 1} \frac{y}{\mu P} = \lim_{\lambda \to 1} \frac{3}{4} \frac{C e^{-b\xi}}{a^2 b} (\operatorname{ch} b\xi + \operatorname{sh} b\xi) \equiv 1 \text{ for all values of C,} \\ &\lim_{\lambda \to 1} \frac{M}{Pl} = - \lim_{\lambda \to 1} \frac{1}{4} \frac{e^{-b\xi}}{b} (\operatorname{ch} b\xi - \operatorname{sh} b\xi) \equiv 0 \text{ for all values of C.} \end{aligned}\right\} \quad (25)$$

For this value of λ, it will be seen that as regards the deflection, at all values of C and for $\xi_p = 0$ as also for $\xi_p = \frac{1}{2}$, the results of the disconnected method correspond completely with those of the continuous method; this is not so in the case of the bending moments.

10. Conclusions and final notes.

The method of calculation, which we have set out, can be simplified in its application to enable us quickly to determine, in the case of any combination of forces applied to the rails, the various deflections and bending moments. For this purpose, it is

necessary to calculate numerically for other values of C and λ the deflections and moments, shewn in Tables V^a-VII and figures 10^a-12^b. It is necessary also to take into account the reaction forces of the bearings. To avoid useless work, however, we consider it desirable to wait experience of cases showing the limits between which the critical limits of C and λ vary.

Since these are unknown, it is not possible to foresee the greatest error which can be introduced by the use of the continuous method. However, allowing that in general C does not exceed a value of 2, it can be expected that, as regards deflection, the differences will never exceed a total of about 10 $^{o}_{o}$ when we follow the method of comparison of flexible constants described in Chapter **8.**

[656 .232 (485)]

Marginal costs – a 40-year old management tool for rate fixing on the Swedish State Railways,

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INTRODUCTION.

Ever since 1910, special calculations of marginal costs of general increases in traffic have been used in Swedish rate-investigations as a guidance for estimations of the profitability of variations in the rates of various kinds of goods. These marginal costs have been termed *minimum rates*.

Minimum rates of the State Railways were first calculated by the Committee on the railway rates of 1907 and were chiefly based upon the figures of traffic and costs of 1910. As these rates, tabulated and given in öre per 100 kg over various kilometre distances, however, correspond only to part of the operating costs of the railway, i. e. costs varying with traffic (variable costs), freight rates must be fixed so much above minimum rates that the freight revenues from all of the tariffs cover also the rest of the railway costs regarded as constant (fixed). Thus minimum rates are no rates in the proper sense of the word, applicable for customers of the railway, but correspond only to certain tabulated costs varying with traffic volume and considered necessary to know for the adjusting of freight rates.

The principles for calculations of minimum rates laid down by the above mentioned Committee were in the main followed by the Board of Management, when the tables of rates of 1929 were prepared and when in 1937 an investigation was made, which should form the basis for the work of the Committee on the railway rates of 1938. The calculations of rates performed by the Committee of 1938 were in the main analogous to the investigations of 1937 and differ as to the method from those of the Committee of 1907 only in respect to costs of maintenance of way, some of which were considered variable by the Committee of 1907, while all these costs in the depression of the 1930s were regarded as constant by the Board of Management and the Committee of 1938.

The minimum rates intended to represent the additional costs arising from increases in traffic. The earlier calculations of these rates have all proceeded from an analysis of the accounted costs of operation and maintenance of a certain year considered representative of a future traffic period. According to the cost categories of the financial accounting and of specially collected particulars the costs of that « normal-year » have been classified as constant (fixed) or variable. While in the calculations constant costs have been disregarded, variable costs

have been assumed to be proportional to certain factors of traffic *(operating quantities)*, such as gross ton-kilometres, car-kilometres, and number of carload shipments. In this way a number of variable average or *unit costs* have been secured by division of the sums of the variable costs of every one of different service and working units by the proportionality factors settled for them. The more the assumption of proportionality between operating quantities and variable costs is true as to traffic increases, the better the calculated unit costs will correspond to the additional costs of the increase.

In earlier calculations of minimum rates the unit costs were grouped under the titles of train, switching, car, and station costs and further they were divided into *terminal costs*, i. e. costs depending upon number of shipments and their character, and *route costs*, i. e. costs depending upon haulage distance. Terminal costs were composed of station costs and part of switching and car costs, while route costs comprised train costs and part of switching and car costs. Starting from the unit costs and special assumptions of average traffic structure as to empty haulage percentage (per cent empty to loaded car-kilometres) and empty car percentage (per cent dispatched empty to loaded cars), marginal costs (minimum rates) of haulage by different types of cars, of various loading weights, and over various distances, were worked out.

The Committee on the railway rates of 1938 calls special attention to the fact that the Committee of 1907 presupposed that the increase in traffic was of a certain moderate extent and that the additional costs changed in the main proportionally to the increase. The report of the Committee of 1938 says on these questions: — In calculating minimum rates, the Committee on the rates of 1907 presupposed that the increase in traffic was of a certain moderate extent, thus neither too small nor too great. In the former case, operating costs were not expected to be affected and in the latter they were affected too much, so that some of the costs, otherwise considered fixed, might increase too. Further the Committee presupposed that the additional costs changed in the main proportionally to the traffic increase, which found expression in the way calculations were performed : after thorough analysis of their characters all the operating and maintenance costs were first divided into variable and fixed (constant), after which the former were put in relation to certain traffic units, such as ton-kilometres, car-kilometres, switched cars, dispatchings, etc.

These fundamental assumptions for the calculation of minimum rates were even at that early stage of a too general character and implied an approximation, as the Committee also pointed out. They could, however, under the circumstances be considered practically valid. In fact, the most important rate-political question was how far certain carload rates could be lowered. Therefore there was no great importance attached to calculations of actual additional costs, but it was deemed sufficient to assume that costs considered variable increased proportionally to traffic. As variable items of costs of practically relevant changes in traffic in most

cases vary considerably more slowly than the traffic volume, the just mentioned assumption of proportionality between costs and traffic volume will as a rule mean that the additional costs are overestimated. Minimum rates worked out on the basis of that assumption must therefore in most cases be regarded as *upper limits* of the additional costs of a traffic change. Yet the Committee of 1938, as also the Board of Management, has considered that rates thus calculated « can serve as guides for the fixing of rates and especially for judgements of reasonable relations between rates of various loading weights ».

Fundamental elements of cost.

For pre-calculations, where, as a rule, the purpose is to get an idea of the probable costs, on special *assumptions of technique and prices*, of a certain quantitative and/or qualitative change in certain traffic achievements, a characteristic of the *variability* of different items of cost under given conditions, is indispensable. The following cost definitions have therefore been introduced by the Railway Cost Committee of 1942.

Some categories of cost vary or can be supposed to vary continuously with changes in production volume. These are called *continuous costs* and can be *proportional*, *progressive* or *degressive* as the relative change in actual costs equals, is greater than, or smaller than the relative change in production volume. Other categories vary by jumps when production passes certain points. These are called discontinuous costs (« jump » costs). Every category can be assigned to one of those two principal types. Under the term of *variable costs* of an assumed production change of defined extent and direction, performed in a certain use of capacity after a certain period of preparation and referring to a certain time, the categories, that vary *under assumed conditions*, are summed up i. e. continuous costs plus certain « jump » costs. The rest of the « jump » costs are on the other hand under these conditions *constant*, i. e. no change arises in those categories.

These definitions, continuous costs and « jump » costs respectively variable and constant costs, will be used in the following. It seems appropriate to introduce special terms to characterize generally the qualities of the cost categories from the « point of view of variability », and other terms to indicate if the categories vary in a certain situation or not.

By total *additional cost* respectively *total subtractive cost* is referred to increase respectively decrease in the *total* costs of a railway enterprise at a certain closely defined increase or decrease in the traffic of one or more traffic branches. It should be observed that by additional cost of a certain change in the traffic of one branch, e. g. merchandise traffic (LCL), it is referred to an increase in the *total* costs of the whole traffic of a railway — thus of all the traffic branches on account of the change in the branch concerned (LCL). In fact, because some of the means of production are used in common, the traffic extent of one branch affects in a degree also items of cost directly dependent upon other branches, in freight as well as in passenger traffic.

If the change in cost arising from variations in a certain achievement is divided

by a factor of the same variation (e. g. gross ton-kilometres), the *additional* respectively *subtractive cost* per unit is obtained. This unit cost of a very small traffic variation is called *marginal cost* (« upwards » respectively « downwards »).

The size of the additional respectively subtractive cost of a certain change will depend upon :

1. The use of capacity that would prevail during the relevant period, if the proposed traffic change should not occur;

2. The direction of the change (increase or decrease in transport achievements);

3. The size of the change at different times;

4. The duration of the change;

5. The duration of the preparatory work for the change.

It can generally be said that the additional cost per unit of a traffic increase will be greater than the subtractive cost of the corresponding decrease.

As a rule it can further be said about the additional cost per unit :— The greater the use of capacity before the change, the greater the change, and the shorter the duration of and preparation for the change—the greater the additional unit cost.

A moderate increase in e. g. merchandise traffic may be met without using an increased number of cars or staff, while a greater increase may require an increased number of cars and staff and, if the increase is very great, also an increased number of locomotives, larger freight house rooms, etc. Additional costs will of course be greater in the last case.

As to the effect of the initial use of capacity on additional costs, the rule is that an increase based on an already high use of capacity may require an increase in locomotives, cars, etc., otherwise not necessary. Additional costs will in this case be greater than if the initial use of capacity had been low.

If the unused capacity of a production element is greater than what is necessary for taking up a certain increase, the costs of the very disposal of that element are constant except for additional depreciation caused by augmented use of the materials.

The duration of the change affects also the disposals necessary to adjust the activity to the traffic variation and thus also the size of additional costs. A short increase must as a rule be taken up by a more intense use of rolling stock, although this requires a considerable rise in need of staff and in empty haulage; a longer, equally great, increase can on the other hand often be met more advantageously by procuring more cars and perhaps by expanding freight houses; by those actions empty haulage and need of staff can be reduced.

As to the effect on additional costs of the duration of preparations there is the rule : — The longer the period of preparation — the more advantageous the carrying out of the change and the smaller the additional cost.

General methods of present calculations of additional costs.

The present calculations of additional (marginal) costs differ fundamentally

from earlier ones performed by the Committees on the rates of 1907 and 1938 and are based upon the general methods of cost calculations outlined in the report of the Railway Cost Committee of 1942 (*). The cost calculations for rate purposes aim at a comparison of the costs in two different situations, i. e. one future hypothetic situation implying a certain quantitative and/or qualitative change in traffic compared to a certain initial situation generally formed by a real traffic situation.

The cost calculations for both situations are established on a so-called *operating planning*, resulting in an *operating plan* comprising route, car, train, and locomotive plans and indicating how transportations shall be performed in actual situations by contributions from the different means of production of the railway. When the initial situation of the change is an actual one, the drawing up of a special operating plan is of course not needed, as there already exists one applied to that situation.

The operating plan of the new situation forms the basis for the calculation of the costs, which calculation is made in two different ways, partly as a *budgeted cost determination*, implying that a budget of the new situation is drawn up for every cost unit of the railway, partly as a *statistical cost calculation* for the new situation.

The budgeted cost determination is supposed to be made in about the same way as the annual budget project for the budget-year. For these calculations are presupposed the creating of partly a central planning organ of the Board of Management, partly regional planning organs of the different regional authorities (districts and sections).

When the costs of a certain traffic situation are to be estimated, the central planning organ gives the regional organs detailed information and data about operating plans, salaries and wages, and material prices to be applied to the new situation. On the basis of that information the regional organs draw up budgets of cost for their spheres of activity with reference to operating and maintenance costs in the new situation. With guidance of data of traffic for the new situation received from the central planning organ, the regional organs must, however, also try to estimate if the capacity of the assets in the planning region is sufficient for the new traffic. This so-called *capacity analysis* performed in co-operation with the central planning organ reveals the need of investments in the new situation.

This method, where the relevant operating quantities are determined on the basis of the operating planning, must as a rule be used for calculations of costs in a new situation. It is then presupposed that route, car, train, and locomotive plans are established in conjunction with route regulations, freight transport regulations, service time-tables, locomotive schedules, etc., already in use. But if variations are small, e. g. about 5-10 % in relation to the initial situation, they can be expected to fall within the frame of

(*) *Principles and Methods of Cost Calculation in the State Railways* (Government Investigation 1949 : 5), Stockholm 1949.

ARNE SJÖBERG : *Production and Cost Structure of the State Railways* (Government Investigation 1949 : 14), Stockholm 1949.

route, car, train, and locomotive plans established for that situation, and then planning measures will often be limited to slight modifications of one or some of the plans.

By means of a simplified method built on certain assumptions of the traffic structure, it will in such cases be possible to reach, without a complete operating planning, a satisfactory determination of the relevant operating quantities of a new traffic situation.

This simplified method is built partly on the assumption of a *constant structure of connections* for the branch of traffic under examination, i. e. it is presupposed that the carried quantities increase or decrease in all connections by the same percentage in relation to the initial situation, partly on the assumption of a *constant transportation method*, i. e. it is presupposed that the freight is in the new situation carried by the same type of car, over the same distances, and by the same type of train as in the initial situation. Both these special assumptions are comprised in the term of *constant transportation structure*.

According to statistical examinations, the deviations of the actual development from those assumptions are in most cases likely to be so small that their effect on costs can be disregarded, and yet an accuracy can be secured that is quite enough for practical purposes. In using this method, no regard is paid to possible transfers between different types of trains and cars. But it is possible to pay attention to such changes in the use of trains and cars as may follow from variations in the haulage order. Rationalizations of transportation method possible in reality mean that actual additional costs can be expected to be lower than those calculated by this simplified method, the application of which may thus cause actual costs to be over-estimated.

The traffic volume and cost conditions of 1945 of the State Railways form the initial situation of the additional cost calculation presented here (*). *A uniform, general increase of 10 % is assumed to occur in carload traffic*. The traffic volume of other traffic branches : passenger traffic, merchandise traffic, etc., is on the other hand presupposed to be unchanged. There should first be worked out average additional costs per 100 kg (carload freight) of this increase, divided into a terminal and a route part, and then there should be arranged an additional cost table, corresponding to previous minimum rate tables and indicating additional costs per 100 kg of carload freight over various distances. As the increase is assumed to be only 10 %, the transportation structure will be considered constant. It will thus be possible to reach, without a complete operating planning, a satisfactory determination of the relevant operating quantities of the new traffic situation. In order to make it possible to arrange the additional costs over various distances in a *table*, when calculations refer to a change in the volume of *one* traffic branch only, there must be introduced special assumptions of the dependence of terminal and route costs

(*) A summary of a study by the present author printed as part 2, *Some examples of marginal cost calculations*, of ARNE SJÖBERG : *Production and cost structure of the State Railways* (Government Investigation 1949 : 14), Stockholm 1949.

upon number of shipments respectively haulage distance. In this calculation, it is therefore assumed partly that terminal costs are proportional to number of carload shipments, partly that route costs irrespective of the part formed by switching costs are proportional to haulage distance.

The additional costs to be worked out on those assumptions will be considered a satisfactory expression of the actual additional costs of this uniform increase. Besides it will be possible to use them as an approximation to additional costs arising from a general but not uniform increase in carload traffic, varying between e. g. 5 % and 15 %, but on an average amounting to 10 %.

If, concerning general changes in traffic, uniform or not, more exact statements of marginal costs should be necessary, a complete operating planning must be undertaken. It will then be possible to calculate only the *total* additional costs of different changes in the extent of the haulage order in various connections (haulage distances), etc., but it will not be possible to arrange the additional costs in tables (minimum rate tables) as above giving costs over various haulage distances. When special (not general) changes in the haulage order are concerned, the cost calculation must likewise be based upon an operating planning, by which the total additional costs of the change in view are determined.

The present calculation of additional costs has run like this. In conjunction with the distribution of operating and maintenance costs between offices, etc., according to the financial accounting it has been examined, account for account, if costs can be expected to be variable as far as this traffic increase is concerned. This judgment of the variance of costs has mainly been founded on regression-statistical examinations. No budgeted cost determination has thus been made by regional planning organs.

Unlike previous calculations, this one is totally based upon *present* prices, i. e. the salaries and wages and material prices of 1945 and that year's cost of procuring such durable means of production, like rolling materials, etc., as can be considered to cause variable costs when they are used in the production. In previous calculations and accounting depreciation costs (interest and depreciation) of durable means of production were calculated on the past costs of procuring them.

From this additional cost calculation the total additional cost of the increase is obtained as a sum of a number of *partial additional costs* of different offices, etc., according to the system of the financial accounting. On the basis of i. a. regression-statistical investigations it is possible to estimate if and in what way the change in traffic makes the different items of cost vary. Such items as cannot be expected to increase at all (or perhaps only slightly) are to be looked upon as *constant* as far as this increase is concerned. The remaining items, the costs of which will change, make *variable costs* of the change in view.

It may be worth mentioning that about the same items will be variable in this way as would have been the case in the minimum rate calculations of the Com-

mittee on the railway rates of 1907. In both calculations way maintenance costs and depreciation costs of rails are thus variable, although those items do not vary in direct proportion to traffic but considerably more slowly. As to way costs, there is thus an important difference between the calculations of the Committee on the railway rates of 1907 and those of the Committee of 1938.

The fundamental assumptions of the Committees of 1907 and 1938 on the *proportionality* of *variable* items of cost to different operating quantities have only exceptionally been used in this calculation. The relationship between costs and operating quantities, so fundamental for the whole calculation of additional costs, have instead in many cases been determined by regression-statistical analyses of material already existing.

The minimum rate laid down by the Committee of 1938 was a weighed average cost intended to be applicable to average traffic conditions of the whole system and was differentiated for carload freight only with regard to type of car, box or flat, and weight of load, 2.5, 5, 10 or 15 tons per car. So there was no differentiation of the minimum rates with regard to different operating powers (electric or steam operation) and train classes (local or through freight train). Such average costs must of course be of limited value, when it comes to special haulage cases differing in one respect or other from average conditions upon which the rate is based, and in reality most cases probably differ considerably from those conditions.

It is therefore important that *minimum rate calculations* should be *more differentiated* than they were previously and this circumstance has been observed here. The calculation of the previously mentioned partial additional costs is mainly based upon regression-statistically defined functions of the relationship between costs and operating quantities in working units as homogeneous as possible as to operating and traffic-technical conditions. For there would be no sense in working out statistical relationship functions for any but homogeneous units. If the additional cost calculation is drawn up like that, a great number of relationship functions and partial additional costs are obtained and a far-going differentiation is possible as to additional costs of different operating powers (electric or steam operation), unequal gauges (standard- or narrow-gauge), lines of unequally high traffic-intensity (i. e. heavy or light traffic), unequal empty haulage percentage, etc. It will thus be possible to lay down a great many *formulas for additional costs* corresponding to the practically relevant combinations of operating and traffic-technical conditions of reality.

The purpose of this examination of additional costs is, however, not to work out all this catalogue of formulas but to give an example *of a way* of *calculating the additional costs* of a certain combination of operating and traffic-technical conditions, in this case the *heavily operated electric main lines* and the *large stations* with specialized distribution of work.

The setting of primary cost determining operating quantities can as to these small variations in traffic volume, for which the above mentioned assumption of a constant transportation structure gives

an accuracy quite sufficient for practical purposes, be made in a relatively simple way and without drawing up a complete operating planning.

As the calculation problem is to compute probable additional costs of different traffic achievements, when carload traffic increases by 10 % compared to the level of 1945, it is thus presupposed that the conveyed quantities of carload, *net-tons*, in all connections increase by 10 % in relation to the initial situation. The *net-ton kilometres* of carload also increase by 10 %.

The number of loaded cars dispatched will also grow by 10 %, as, to judge from the traffic experiences of 1943-1945, an increase in carload transportations is very unlikely to find expression in an essentially augmented loading of the number of cars already conveyed. Thus the number of *loaded car-kilometres* will also increase by 10 %.

The number of empty cars dispatched and empty car-kilometres can on the other hand not be secured so easily, but these quantities must be estimated on the basis of available experiences. In the additional cost formulas they constitute *empty haulage percentage*, i. e. per cent empty to loaded car-kilometres, respectively *empty car percentage*, i. e. per cent dispatched empty to loaded cars.

The number of *terminal switchings* and *switchings on route* of cars can in this case also be expected to increase by 10 %. As freight trains can generally be better used when traffic increases, *the number of train-kilometres* will not grow in the same proportion as the number of loaded cars but considerably more slowly. When the number of train-kilometres in the new situation is calculated, the degree of actual use of trains on different lines must be considered.

With regard to their haulage tasks, freight trains are divided into *direct freight trains* (for exchange of cars between two defined stations), *through freight trains* (for exchange of cars between large stations on a certain line) and *local freight trains* (for traffic to and from small stations). Direct and through freight trains, in the following comprised under the latter term, stop at large stations only and are operated over long distances. Local freight trains generally stop at all the small stations, where their locomotive is used for required switching of cars. The train staff assists that of the station with occurring switching and also with unloading and loading merchandise traffic.

Owing to their long stops, the operating speed of local trains is very low, 10-15 km/h, compared to that of through trains, 35-45 km/h.

Local trains are only operated over fairly short distances and their use is mainly limited to such freight as has necessarily to be conveyed by them, while all the rest of the traffic is handled by through freight trains. As a rule far-away cars to or from small stations thus move on local trains only to or from the nearest large station where through trains stop. For that reason the schedules for the main lines contain one regular local train in each direction a day and a number of regular through freight trains adjusted to the need on every particular line.

Train movement is thus adapted to traffic volume by increases or decreases

in the number of through freight trains on the line in view, while the number of local trains is constant. But a traffic increase, measured in gross ton-kilometres or car-kilometres, can naturally cause local trains to increase in length and weight (number of car-kilometres per train-kilometre respectively gross ton-kilometres per train-kilometre). As the number of *local trains* is constant, independent of the extent of traffic, the *costs* of *these trains* are mainly constant, when traffic changes occur. It is therefore justifiable to consider that variable costs of this increase arise only when it is a question of *through trains*. The augmented gross transport work for carload freight, measured in car-kilometres and gross ton-kilometres can be taken up and carried out partly by a greater number of through freight trains (i. e. an increase in train-kilometres of through freight trains), partly by greater weights of trains (i. e. an increase in gross ton-kilometres per train-kilometre of through freight trains).

Statistical examinations of relationships between gross ton-kilometres and train-kilometres on the main lines during the last few years, show that a 10 % increase in gross ton-kilometres causes some 8 % increase in train-kilometres, which implies an increase in train weight of about 2 %.

Additional costs of operation and transportation.

The present calculation of additional costs shows i. a. the difference from that of 1938 that the very distinction of variable items of cost of the actual increase is built on *relationship functions*, where the respective item has been put in relation to the cost determining primary operating quantities (train-kilometres, gross ton-kilometres, car-kilometres, number of dispatched cars, number of shipments [freight waybills], number of attached and detached cars). On the basis of these cost relationships there can, for different cost items, be calculated *partial additional costs per unit of operating quantity, here called additional operating costs*, arising from variations in the extent of operating achievements these being measured in the primary cost determining operating quantities just mentioned.

Starting from these partial additional operating costs together forming the total additional cost of the increase, and after introducing special assumptions of the use of rolling stock, etc., *additional costs per unit of transport service* can be calculated (öre per weight unit of load over various distances). These quantities will be called *additional transportation costs*.

By calculating partial additional costs a better knowledge is obtained of the relative significance of different cost elements in the main groups of cost, than what was previously possible. By this specification of cost elements it is further easy to estimate the effects of changes in the level of salaries and wages and material prices on total additional costs. By the accounting of partial additional costs it will finally, after certain adjustments, in many cases be possible to use the present additional cost figures also for cost calculations referring to traffic changes of short effect. In some traffic situations it may thus be possible to take up small increases in traffic without increases in

movement, staff, or maintenance and depreciation costs of locomotives. In such cases these items of cost are therefore not to be regarded as partial additional costs but as constant ones. Similarly maintenance and depreciation costs of cars may in a certain situation and at a certain traffic increase be considered constant. In that way a considerable differentiation and adaptation of additional cost calculations to conditions in a number of special cases can be performed.

Arranging of additional operating costs.

For calculations of additional costs arising from the traffic variation concerned, it is referred to the memorandum of the author in Appendix 2 of the Report of the Railway Cost Committee of 1942.

In its summing up of calculations of the cost elements here termed additional operating costs the Committee on the rates of 1907 pointed out that from examinations of different items it was evident that railway costs were mainly variable and — within the limits of examination — proportional to one of the following operating quantities.

Number of train-kilometres of different categories of trains;

Number of gross ton-kilometres of different categories of trains;

Number of switchings, terminal or on route, of different types of cars;

Number of car-axle-kilometres (or car-kilometres) of different types of cars;

Number of loaded or empty cars dispatched;

Number of tickets sold;

Number of shipments of registered luggage or of goods of various kinds, or their weight.

In the first place the Committee put together costs considered variable into four main groups, namely train, switching, car, and station costs.

By *train costs* were understood such as were proportional either to train-kilometres or to gross ton-kilometres of different categories of trains; *switching costs* were proportional to number of cars attached and detached; *car-costs* to car-kilometres or to number of cars dispatched, and *station costs* (including costs of printing of tickets, and control and allocation of ticket fares and carriages) to number of tickets sold or to number of freight shipments or their weight.

As these definitions of several categories of unit costs do not seem to make the description clearer, they have here been abandoned except for « *switching costs* ».

With regard to the cost determining operating quantities the partial additional costs have instead been grouped into train-kilometre costs, gross ton-kilometre costs, car-kilometre costs, etc. « Switching costs » are here not used in the restricted sense that they shall necessarily be proportional to number of cars attached and detached. In calculating them it is however, for want of intimate experience of their character, necessary to start from the same simplified assumption as did the Committee of 1907, i. e. that they are proportional to number of cars attached and detached.

The partial additional costs included in *switching costs* are :

	Öre per switching hour
1. Locomotive staff	924
2. Electric power	64
3. Depreciation costs of locomotives	405
4. Compounds for oiling and polishing locomotives	35
5. Maintenance of locomotives	146
6. Engine house service	52
Sum	1 626

	Öre per car attached and detached
11. Station service costs	62.5

As appears from this table switching costs have in the first place been calculated as pure *temporal* costs, except for unit costs of station service under point 11, which have been directly related to number of cars attached and detached.

Number of cars attached and detached per switching hour appears from this table :

	Number of cars attached and detached per switching hour
6 larger stations	42.0
101 stations	14.8
107 stations	18.7

With guidance of these figures it has been considered reasonable to count with 18 cars attached and detached on an average per hour. From that value and the before mentioned partial additional costs the following switching costs are obtained per car attached and detached :

Switching costs per car attached and detached, öre (electric operating).

	Partial additional costs as above
1-6 (Locomotive staff, etc.)	90.3
11 (Station service costs)	62.5
Sum of switching costs	153

Tables of *all the partial additional costs* of electrically operated through freight trains will be found below; these costs having been grouped with reference to operating quantities considered cost determining — train-kilometres, car-kilometres, gross ton-kilometres (of cars), number of cars dispatched (loaded respectively empty), number of shipments of carload freight and number of cars attached and detached.

I. Additional operating costs in öre per train-kilometre. (Corresponding costs in öre per gross ton-kilometre have been put down in brackets.)

1. Locomotive staff	34.6	(0.044)
2. Electric power (*)	6.2	(0.008)
3. Depreciation costs of locomotives	16.6	(0.021)
4. Compounds for oiling and polishing locomotives	3.7	(0.005)
5. Maintenance of locomotives	14.6	(0.018)
6. Engine house service	5.2	(0.007)
11. Train dispatching	20.7	(0.026)
13. Contact-wire	1.2	(0.002)
Sum I	102.8	(0.130)

II. Additional operating costs in öre per gross ton-kilometre (of cars).

2. Electric power	0.014
12. Depreciation costs of rails	0.042
Maintenance, watching and patrolling of way	0.025
Sum II	0.081

III. Additional operating costs in öre per car-kilometre.

	Box car	*Flat car*
2. Electric operation	0.24	0.24
7. Maintenance of car-axles	0.32	0.18
8. Car service and supplies	0.60	0.61
10. Maintenance of car bodies	0.08	0.08
Depreciation costs of cars	0.33	0.18
Sum III	1.57	1.29

(*) Costs of electric power have been assigned partly to I, partly to II and III.

IV. Additional operating costs in öre per car dispatched.

	Box car			Flat car		
	Dispatched loaded	Dispatched empty	Resupplying for new load	Dispatched loaded	Dispatched empty	Resupplying for new load
Maintenance of car body .	157	61	36	193	68	45
Depreciation costs of cars . .	633	243	146	457	162	108
Sum IV . . .	790	304	182	650	230	153

V. Additional operating costs in öre per shipment (freight waybill).

11. Station service costs	273
Staff costs of Control office. . . .	36
Sum V.	309

VI. Additional operating costs in öre per car attached and detached.

1-6 and 11. Switching costs	153

Calculation of additional transportation costs on the basis of additional operating costs and certain special assumptions.

The special assumptions required to make it possible to calculate, on the basis of additional operating costs, the final additional costs of the very *net* transportation work (öre per weight unit of load carried over various distances) can first be divided into the following three main categories :

1. Assumptions of the use of rolling stock.

2. Assumptions of the number of switchings on route over various distances.

3. Assumptions of the number of cars per shipment.

The assumptions of the use of rolling stock can be grouped in these two main categories :

1*a*. Assumptions of the use of traction power of trains.

1*b*. Assumptions of the relative dispatching and haulage of empty cars (empty car percentage, empty haulage percentage).

1*a*. *Use of traction power.*

The increase in gross ton kilometres (x) arising from the traffic increase is denoted by Δx, and the increase in train-kilometres (y) by Δy. The additional operating costs according to point I of the preceding account (train-kilometre costs) can be « transformed » to additional operating costs counted per gross ton-kilometres by multiplying train-kilometre costs by the quotient of $\frac{\Delta y}{\Delta x}$.

The quotient of $\frac{x}{y}$

$$\left(\text{i. e. } \frac{\text{gross ton-kilometres}}{\text{train-kilometres}}\right)$$

gives the average weight of trains. If it

is assumed that there is direct proportionality between train-kilometres and gross ton-kilometres — the average weight of trains is in other words assumed to be unchanged, although traffic increases — then $\frac{\Delta y}{\Delta x}$ will equal $\frac{y}{x}$ (= the inverted value of the average weight of trains). It has, however, in this analysis not been presupposed that the average weight would be unchanged, but that the traffic increase would partly be met by an increase in the (average) weight of trains. The quotient of $\frac{\Delta y}{\Delta x}$ is thus supposed to be smaller than $\frac{y}{x}$.

For the calculation of the quotient of $\frac{\Delta y}{\Delta x}$ the relationship between the operating quantities of train-kilometres and gross ton-kilometres of through freight trains have been examined for several electric lines during 1935-1946. The regression equation $y_n = a_n + b_n x_n$ denoting the relationship between train-kilometres (y_n) and gross ton-kilometres (x_n) on the line n, has been worked out for several lines on the basis of statements of train-kilometres and gross ton-kilometres of different years. The obtained regression coefficients b_n have been arranged in the table on the next page in which the length of the lines and the quotient of $\frac{\text{train-kilometres}}{\text{gross ton-kilometres}}$ of 1945 (= the inverted value of the average weight of trains in 1945) are also given.

To secure an average value of the regression coefficients of different lines the former have been weighed together, partly with train-kilometres, partly with gross ton-kilometres as weights, and in both cases *the average value* has shown to be $b = 0.00126$, which in the present analysis has been used as a value of the quotient of $\frac{\Delta y}{\Delta x}$ for the transformation of train-kilometre costs into gross ton-kilometre costs. As a comparison it can be mentioned that, if direct proportionality had been assumed between train-kilometres and gross ton-kilometres, the quotient just mentioned would have amounted to 0.00150, the average weight of the trains being that of 1945 (667 tons). This value is 19 % higher than the one calculated on the basis of the regression analysis (0.00126).

Applying the value of $b = 0.00126$ *the transformation of train-kilometre costs into gross ton-kilometre costs* gives a cost of ($0.00126 \times 102.8 =$) 0.130 *öre per gross ton-kilometre*.

1b *Empty car and haulage percentage.*

For the calculation of additional transportation costs there must also be introduced certain assumptions of empty haulage of cars. The percentage of empty to loaded car (—axle)-kilometres is generally called *empty haulage percentage*.

From train reports are obtained statements of car-axle kilometres of *individual lines*, which are specified monthly for box and flat, loaded and empty cars. In this way empty haulage percentage can be calculated per month and line for box and flat cars.

The Committee on the railway rates of 1938 calculated with an empty haul-

age of 40 % (for box as well as flat cars). The average empty haulage of the whole system amounted in 1938 to 44 % and in 1945 to 32 %. In August 1945 the percentage was 40 for flat cars and 15 for box cars.

When calculating additional costs according to the formulas for additional transportation costs laid down in the following, there can be introduced alternative assumptions of increase in empty haulage, given in car-kilometres of empty cars, caused by an increase in carload

LINE	Operated length of line in 1945 km	*Train-km* gross ton-km in 1945	Regression coefficient b_n	Year of observation of b_n	Opened to electric operation
Stockholm C-Hallsberg	201	0.00158	0.00132	35—46	26
Hallsberg-Falköping C	144	0.00154	0.00120	35—46	26
Falköping C-Göteborg C	114	0.00155	0.00126	35—46	26
Järna-Mjölby	195	0.00165	0.00157	35—46	32
Mjölby-Hässleholm	274	0.00144	0.00118	35—46	33
Hässleholm-Malmö C	83	0.00189	0.00170	35—46	33
Krylbo-Mjölby	253	0.00140	0.00113	35—46	32—34
Katrineholm-Åby	41	0.00192	0.00144	35—46	32
Falköping C-Nässjö	113	0.00161	0.00138	35—46	32
Malmö C-Halmstad C	148	0.00164	0.00108	36—46	34—35
Halmstad C-Göteborg C	152	0.00156	0.00093	37—46	36
Charlottenberg-Laxå	212	0.00157	0.00120	38—46	37
Stockholm C-Krylbo	161	0.00144	0.00131	35—46	34
Krylbo-Ljusdal	220	0.00133	0.00123	37—46	35—36
Ljusdal-Ånge	106	0.00144	0.00128	37—46	36
Ånge-Bräcke	31	0.00134	0.00120	40—46	39
Bräcke-Östersund C	71	0.00173	0.00139	40—46	39
Bräcke-Långsele	131	0.00125	0.00116	40—46	39
Uppsala C-Gävle C	132	0.00165	0.00143	38—46	37
All electric lines (*)	4 595	0.00150			

(*) Average length of electrically operated lines in the year.

traffic, given in car-kilometres of loaded cars. The quotient of increase in kilometres of empty cars and increase in kilometres of loaded cars, given in per cent, is in these formulas denoted by p %.

As has been mentioned above, the quotient of number of empty and loaded cars dispatched was also among the assumptions for the calculation of minimum rates of the Committee of 1907. This quotient (expressed in per cent) is called *empty car percentage*. Between empty haulage percentage, empty car percentage and average haulage distances of loaded and empty cars respectively, there is the relation :

$$\frac{\text{Empty car percentage}}{\text{Empty haulage percentage}} = \frac{\text{Average haulage distance of loaded cars}}{\text{Average haulage distance of empty cars.}}$$

For the definition of empty car percentage and average haulage distance of empty cars, it is appropriate to start from such a definition of the haulage of an empty car that the whole course from the dispatching of the car from the unloading station to its arrival at the new loading station is comprehended in it.

As when empty haulage percentage is concerned, it is in this calculation of additional costs possible to count with alternative values of empty car percentage, in the formulas below denoted by q. The values of the quantities of p and q should of course be chosen so that the quotient of $\frac{q}{p}$ will be consistent with the mentioned « empty car formula ».

2. *Switchings on route.*

By *terminal costs* are in these calculations understood variable costs arising in connection with treatment of the freight partly at the dispatching station (until the car is attached to the train), partly at the destination station (from the point when the car is detached from the train). Remaining variable costs are regarded as *route costs*. Of additional costs mentioned in the tables of partial additional operating costs, train-kilometre, gross ton-kilometre, and car-kilometre costs (point I-III) are classified only as route costs, while all the car-dispatching and shipment costs (point IV and V) are terminal costs. Switching costs (point VI) are, on the other hand, included in both route and terminal costs.

The last-mentioned costs arise partly where a car is dispatched and received (terminal handling), partly where it is transferred to another train on its way to the destination station. The former are terminal costs, while the latter, which increase with haulage distance, are route costs.

The Committee of 1938 assumed like that of 1907 that number of on route car-transfers was directly proportional to length of way. For loaded cars (box and flat) was assumed one transfer (*) for every 163 kilometres, i. e. 0.00614 transfers were assumed to occur for every car-kilometre. For empty cars, on the other hand, transfers were assumed to occur only for every 294 kilometres, corresponding to 0.00340 transfers per kilometre. (The lower frequency for empty cars depends upon the committee's using another definition of empty haulage than the one applied here.)

(*) One transfer corresponds to two switchings on route (one detachment and one attachment).

Haulage distance km	Number of observed loaded runs	Number of transfers per car	Number of transfers per km (and car) (*)
1 — 9	744	0.09	0.018
10 — 19	1 081	0.32	0.021
20 — 29	567	0.25	0.010
30 — 39	565	0.35	0.010
40 — 49	927	0.44	0.010
50 — 59	608	0.50	0.009
60 — 69	501	0.97	0.015
70 — 79	356	0.64	0.009
80 — 89	267	0.67	0.008
90 — 99	148	0.80	0.008
100 — 149	1 310	1.19	0.010
150 — 199	858	1.32	0.008
200 — 299	1 178	1.64	0.007
300 — 399	738	2.19	0.006
400 — 499	564	2.29	0.005
500 — 599	308	2.39	0.004
600 — 699	186	2.70	0.004
700 — 799	91	3.27	0.004
800 — 899	105	3.20	0.004
900 — 999	51	3.35	0.004
1 000 — 1 499	211	3.45	0.003
1 500 —	11	4.27	max. 0.003
Sum	11 375	—	—

(*) For the working out of the values in this column there has been calculated with a distance corresponding to the middle of the respective distance intervals (5, 15, 25 km, etc.).

The assumptions of the Committee of 1938 were built on a special investigation of the freight-car movement during the period 15th-31st of October 1934. Number of transfers (and number of observed forwardings of cars) was reported distributed among different haulage distances as is shown in the table above.

Those observations of 1934 can apparently not be interpreted so that number of transfers would be directly proportional to length of way. The value used by the Committee of 1938 — about 0.006 transfers of loaded cars per kilometre — which corresponds to the ratio between total number of transfers and number of car-kilometres of observed runs of loaded cars, causes an overestimation of number of transfers for distances over 400 kilometres and an underestimation for distances under 300 kilometres.

In the following formulas for additional costs number of switchings as a function of haulage distance x (kilometres) has been denoted by S (x). The actual relevant assumption of the number of transfers can thus be introduced into the practical calculations.

3. *Number of cars per shipment.*

The ratio between number of dispatched loaded cars and number of shipments (dispatchings or freight waybills) is relevant for the calculation of dispatching costs (cf point VI in the following table). A calculation of this quotient has been made for April 1947 on the basis of the primary material of current statistics of carloads.

It is mainly when it is a question of bulk freight that a shipment refers to several cars. An arrangement of different goods groups according to number of cars per shipment gives the following table. (See next page.)

At present there are no examinations available of the increase in dispatching costs, when the number of cars per shipment increases. Experience shows, however, that these costs increase very slowly in such cases. According to present regulations 5 cars can at the most constitute one shipment, i. e. be put down on the same freight waybill.

With regard to the relative constancy of dispatching costs, when number of cars per shipment is changed, these costs can in the calculations be assumed to be equally great of all carload shipments independent of number of cars dispatched.

In the formulas for additional costs number of cars per shipment has been denoted by n.

4. *Formulas for marginal transportation costs.*

Besides the assumptions under 1-3 above of the use of the capacity of trains and cars, statements of tare weight of used cars are also required for calculations of marginal (additional) transportation costs. As calculations of additional operating costs have at least in principle been based upon a model of calculation, where only standard cars have been presupposed to be used and procured to meet the assumed traffic increase, it follows that *tare weights* of standard cars, namely *14 tons for box cars* and *11 tons for flat cars*, must be used also when additional transportation costs are calculated. These data are to be regarded as

technical constants and are logically to be classified to quite a different category than figures forthcoming from the assumptions 1-3 above. In building up minimum rates both the Committee of 1907 and that of 1938 used *average* tare weights obtained from observations of cars in traffic. The latter Committee thus proceeded from a tare weight of 10.9 tons of box cars and 9.2 tons of flat cars.

The formulas below for marginal transportation costs refer to carload transportations with a loading weight (net weight) of w tons per car. Actual loading weights

GOODS GROUP	Number of dispatched cars	Number of shipments	Number of cars per shipment
Iron Ore (not of Lapland)	3 119	681	4.6
Paper pulp	2 874	1 100	2.6
Stone, crude and minerals	2 743	1 348	2.0
Saw chips, wood-wool, etc.	1 549	836	1.9
Matches	109	61	1.8
Mineral coal and coke	4 624	2 646	1.7
Pig-iron, etc., and scrap	1 920	1 113	1.7
Paving-stone	295	188	1.6
Various wares	992	630	1.6
Various vegetable products	218	141	1.5
Peat and peat products	830	564	1.5
Sugar, syrup	749	505	1.5
Earth, clay, gravel, sand, building lime	2 815	1 847	1.5
Forgeable iron, plate, roller thread	3 583	2 650	1.4
Mineral oils (burning and lubricating)	3 808	2 828	1.3
Rubber and rubber products	237	183	1.3
Other metals and metal works	1 224	912	1.3
Paper wood and pitprops	2 640	2 067	1.3
Pasteboard, incl. fibre plates, paper	5 450	4 332	1.3
Other goods groups	.	.	less than 1.25(*)

(*) Except for transportations for the defences and haulages of railway vehicles.

per car can thus be introduced into the formulas.

In practical use of these formulas meant for rate purposes, there should in the first place be calculated marginal costs of the lowest weights to which the different tariffs of 2.5, 5, 10 and 15 tons are applied, viz. 2.5, 5, 10 and 15 tons per car respectively.

The following symbols will be used in the formulas for marginal costs :

Haulage distance	x km
Loading weight per car	w tons
Empty haulage percentage . . .	p %
Empty car percentage	q %
Number of switchings over a haulage distance of x km	S (x)
Number of cars per shipment . .	n

Marginal costs in öre per car

Terminal costs of box cars . . .	$T_b\ (q, n)$
Terminal costs of flat cars . . .	$T_f\ (q, n)$
Route costs over a haulage distance of x km of box cars . .	$R_b\ (w, x, p, q)$
Route costs over a haulage distance of x km of flat cars . .	$R_f\ (w, x, p, q,)$

The different groups of partial marginal costs forming terminal respectively route costs of haulage of a car over a distance of x km have been arranged in the following tables.

From these tables there can be derived the following formulas (see page 71) for calculating *marginal costs per car* of the State Railways of terminal handling respectively haulage of *carloads* in box or on flat cars over a distance of x kilometres by *electrically operated through freight trains.*

Terminal costs in öre per car.

Marginal costs according to pages 62-63	Box car	Flat car
IV. Car dispatching costs : loaded car	790	650
empty car, dispatched	$\frac{q}{100} \cdot 304$	$\frac{q}{100} \cdot 230$
empty car, re-supplied for new load . .	$\left(1 - \frac{q}{100}\right) \cdot 182$	$\left(1 - \frac{q}{100}\right) \cdot 153$
V. Shipment costs	$\frac{309}{n}$	$\frac{309}{n}$
VI. Switching costs (terminal switching) : loaded car	$2 \cdot 153$	$2 \cdot 153$
empty car	$\frac{q}{100} \cdot 2 \cdot 153$	$\frac{q}{100} \cdot 2 \cdot 153$
Sum of terminal costs	$T_b\ (q, n)$	$Tf\ (q, n)$

Route costs in öre per car over a haulage distance of x km.

Marginal costs according to pages 62-63	Box car	Flat car
I. Train-km costs (*) :		
loaded car	$(w+14) \cdot 0.130 \cdot x$	$(w+11) \cdot 0.130 \cdot x$
empty car	$\frac{p}{100} \cdot 14 \cdot 0.130 \cdot x$	$\frac{p}{100} \cdot 11 \cdot 0.130 \cdot x$
II. Gross ton-km costs :		
loaded car	$(w+14) \cdot 0.081 \cdot x$	$(w+11) \cdot 0.081 \cdot x$
empty car	$\frac{p}{100} \cdot 14 \cdot 0.081 \cdot x$	$\frac{p}{100} \cdot 11 \cdot 0.081 \cdot x$
III. Car-km costs :		
loaded car	$1.57 \cdot x$	$1.29 \cdot x$
empty car	$\frac{p}{100} \cdot 1.57 \cdot x$	$\frac{p}{100} \cdot 1.29 \cdot x$
IV. Switching costs (switching on route) :		
loaded car	$S(x) \cdot 2 \cdot 153$	$S(x) \cdot 2 \cdot 153$
empty car	$\frac{q}{100} \cdot S\left(\frac{p}{q} \cdot x\right) \cdot 2 \cdot 153$	$\frac{q}{100} \cdot S\left(\frac{p}{q} \cdot x\right) \cdot 2 \cdot 153$
Sum of route costs	$R_b(w, x; p, q)$	$R_f(w, x; p, q)$

(*) Converted into gross ton-km costs.

$$T_b(q, n) = 1278 + \frac{309}{n} + \frac{q}{100} \cdot 428$$

$$T_f(q, n) = 1109 + \frac{309}{n} + \frac{q}{100} \cdot 383$$

$$R_b(w, x; p, q) = \left[4.52\left(1 + \frac{p}{100}\right) + 0.211 \cdot w\right] \cdot x +$$

$$306\left[S(x) + \frac{q}{100} \cdot S\left(\frac{p}{q} + x\right)\right]$$

$$R_f(w, x; p, q) = \left[3.61\left(1 + \frac{p}{100}\right) + 0.211 \cdot w\right] \cdot x$$

$$306\left[S(x) + \frac{q}{100} \cdot S\left(\frac{p}{q} \cdot x\right)\right]$$

The marginal costs per 100 kilograms of loading weight of box respectively flat cars, MC_b respectively MC_f, over a haulage distance of x km, can be given in these two formulas :

$$MC_b = [T_b\,(q,\ n) + R_b\,(w,\ x;\ p,\ q)] \cdot \frac{1}{10w} \text{ (box car)}$$

$$MC_f = [T_f\,(q,\ n) + R_f\,(w,\ x;\ p,\ q)] \cdot \frac{1}{10w} \text{ (flat car)}$$

On the basis of these formulas for additional transportation costs it is possible to calculate marginal costs per 100 kg of loading weight for variously combinated assumptions of types of car, loading weight, haulage distance, empty traction percentage, and empty car percentage. The marginal costs for various haulage distances calculated for a certain combination of assumptions — e.g. box car, 2.5 tons, empty traction percentage 40, empty car percentage 70 — can then be grouped together into a table similar to the previous minimum rate tables. As, however, the number of combinations of assumptions is considerable, the collected tables would be very numerous, if all relevant combinations were tabulated. Such a number of tables will not be very surveyable, and it would certainly be fairly time — consuming to obtain the marginal cost for an actual case from them. To make a swift marginal cost calculation in actual cases possible and to facilitate a judgement of the effect on marginal costs of changes in loading weight, empty traction, etc., the final outcome of the calculation has therefore been arranged in *nomograms* referring to terminal and route costs for flat respectively box cars. By help of these nomograms marginal costs per 100 kg of loading weight can be read off swiftly and in a simple way for various combinations of assumptions of loading weight, empty traction, etc.

NOMOGRAM FOR MARGINAL HAULAGE COSTS OF CARLOADS IN 1945.

Rules of interpretation

Leading line = — — — — — — — — —

Auxiliary line = ———————

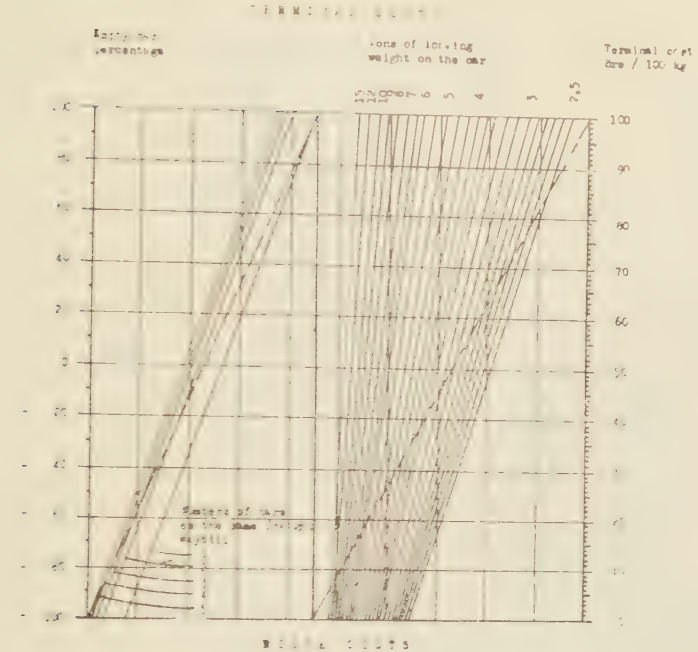
As to both terminal and route costs you should start from the left edge of the left field, go horizontally to the required auxiliary line, vertically to the leading line, and then horizontally to the required auxiliary line of the next field, vertically to the leading line, etc., until you meet the scale of cost in the right edge. When, in that way, terminal and route costs have been determined separately, they should be added up, thus composing total marginal costs in öre per 100 kg of the haulage concerned.

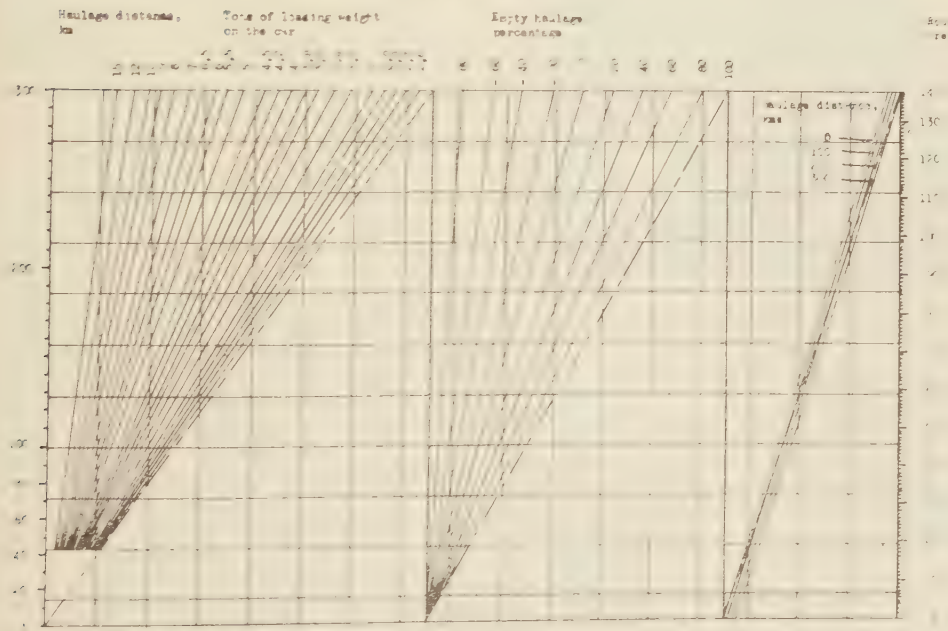
The nomogram for route costs includes haulage distances from 30 up to 300 km but can be used also for cost calculations for transportations over and under these limits. For distances under 30 km the actual distance should be multiplied by 10, the nomogram be read off, and the result divided by 10.

For distances over 300 km the actual distance should be divided by 10, the nomogram read off, and the result multiplied by 10.

Marginal haulage costs of carloads in 1945

BOX CAR

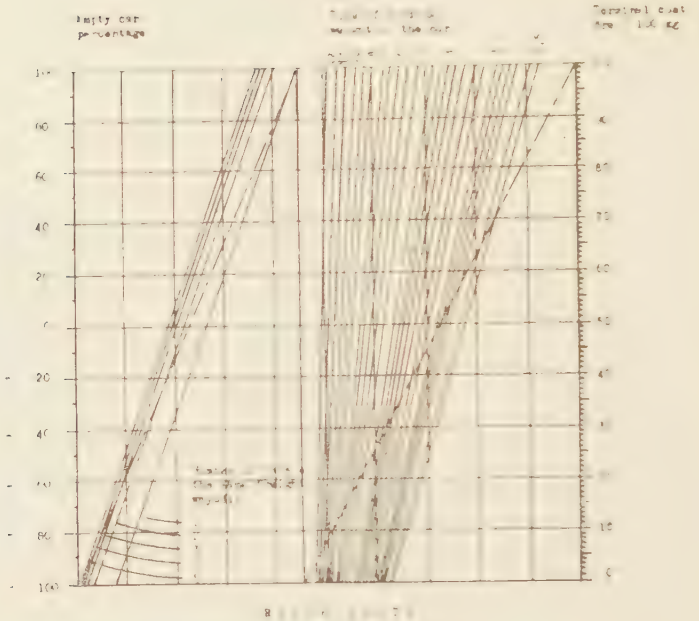


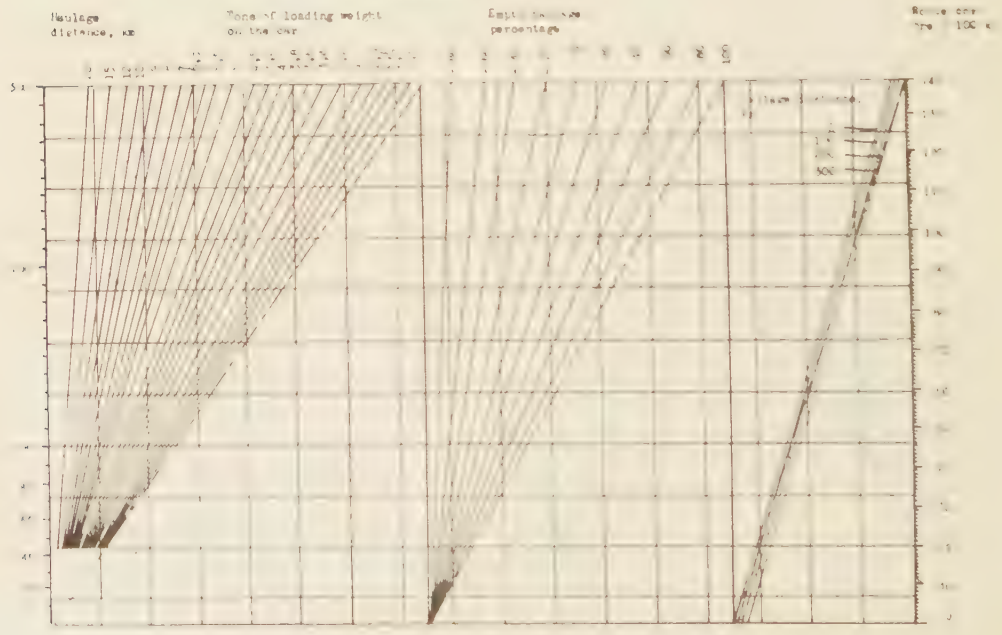


Marginal haulage costs of carloads in 1945

FLAT CAR

TERMINAL COSTS





[385. (09 .2]

OBITUARY.

Axel GRANHOLM,

Former General Manager of the Swedish State Railways.
Former Member of the Permanent Commission of the International Railway Congress Association.

On the 6th July 1951, there passed away at the age of 79 years, Mr. Axel Granholm, former General Manager of the Swedish State Railways and former Member of the Permanent Commission of our Association.

Mr. Granholm was born on the 19th August 1872. He matriculated in 1890 and entered the Stockholm Polytechnic School, where he obtained his civil engineering diploma for Roads and Bridges.

Mr. Granholm entered the service of the Swedish Railways in 1898. This date coincided with the construction of the line from Gällivare-Kiruna to the Norwegian frontier, an extension of the Luleå-Boden-Gällivare line and this work was the beginning of Mr. Granholm's distinguished service of nearly forty years on the Swedish Railways. The construction of this railway line in the isolated mountainous regions of Lapland, away in the Artic regions, certainly represented a difficult task and the way in which he carried it out earned him a great reputation as a skilful and energetic director of labour.

After having directed the construction of other lines during the period between 1903 and 1910, Mr. Granholm was charged with the important task of building the dam and hydro-electric plant at Porjus, which has an underground machinery room. The successful issue of these undertakings brought him to the notice of the Government and added to his reputation. It was therefore no surprise to those who appreciated the qualities of Mr. Granholm as a leader, when the post of General Manager of the Swedish State Railways, which became vacant in 1913, was offered to him. He was then forty one years old, which at that time was considered young to take on a post with such heavy responsibilities.

Already during his first years as General Manager, he had to run the railways at a

time when heavy burdens were placed on railways by the first world war, which was marked by a very great increase in the traffic, a considerable increase in the staff and administrative organization, as well as by constant increases in operating costs. In spite of the troubles and vexations due to the war, he worked energetically, fully conscious of the object to be attained, which was to put the railways on a commercial standing. To do this, he was at constant pains to raise the technical standards, both as regards the permanent way and the rolling stock. With a clear vision of the economic importance of electrification, he did everything possible to electrify those lines suitable for this method of working. Consequently when he retired in 1937, 42 % of the State railways were completely electrified.

Originally Mr. GRANDHOLM had no excessive fear of road motor competition which after 1920 became ever greater and greater. But as the situation became clearer and the Swedish Railways began to feel the need to fight such competition, he did everything possible to equip the railways with motor services on the one hand, by opening up new routes, and above all, on the other hand, by buying out private motor undertakings, and during his management, the Swedish State Railways had an important motor traffic. Not only during his management but also after his retirement, the Government entrusted Mr. GRANHOLM with many vital missions, in connection with transport, the supply of electricity, and administrative organization.

In his railway policy, Mr. GRANDHOLM was fully aware of the importance of maintaining good relations with the other Nordic countries and Central Europe.

Mr. GRANHOLM belonged to those men who contributed by their work to the prosperity of Sweden round about 1900. His life was a rare combination of three conceptions: worker, engineer, and administrator. He devoted himself to the railway with the whole of his physical and spiritual strength. Its prosperity and its trials held the whole of his interest which lasted undiminished until the end of his life.

Mr. GRANHOLM was elected a member of the Permanent Commission of our Association in 1920. He took part in the London Congress in 1925, Madrid in 1930 and Paris in 1937, as a representative of the Swedish Government. In spite of his many activities, he was always a faithful and devoted collaborator of our Association.

We wish to tender our sincere sympathy to his family.

The Executive Committee.

CORRIGENDUM

The following alterations must be made to the article, written by Mr. J. DUBUS, on : « Vierendeel girders : stress analysis, design and welding », appearing in our *Bulletin* for the month of December 1949 :

Page 854 : *a)* The formulae under : *Loading of the top chord* and *Loading of the bottom chord* must read as hereafter :

Loading of the top chord :

$$\text{Axial load } N_r', {}_{r'+1} = -U_r$$

$$\text{Shear } T_r', {}_{r'+1} = \frac{1}{2} T_r$$

$$\text{Bending moment } (M_r', {}_{r'+1})\ x = + \frac{1}{2} (M_x - U_r \cdot h)$$

$$(M_r', {}_{r'+1})x = + W_r + Z_r \cdot x - U_r \times \frac{h}{2}$$

$$= +\frac{1}{2} M_r + \frac{1}{2} T_r \cdot x - U_r \times \frac{h}{2} = + \frac{1}{2} (M_x - U_r \cdot h)$$

Loading of the bottom chord.

$$\text{Axial load } N_r'', {}_{r''+1} = + U_r$$

$$\text{Shear } T_r'', {}_{r''+1} = \frac{1}{2} T_r$$

$$\text{Bending moment } (M_r'', {}_{r''+1})x$$

$$= + \frac{1}{2} (M_x - U_r \cdot h)$$

$$\text{as } (M_r', {}_{r'+1})x + (M_r'', {}_{r''+1})x = M_x - U_r \cdot h$$

b) Add the following note at the bottom of this page :

Note. — In Fig. 2, the formulae $(M_{r\,r-1})x$ and $(M_r', {}_{r''-1})x$ must be altered as follows : $(M_r', r'+1)x$ and $(M_r'', r''+1)x$.

Page 855 : The formula : $W_{r1t} = 0\ Z_{r1t} = 0$ must read $W_r, t = 0\ Z_r, t = 0$.

Page 865 (2nd column) : Add the sign — in front of the following formula :

$$\frac{3\ D(H_r + H_{r+1})}{2\ H^3{}_{r+1}} \cdot M_{r+1}$$

MONTHLY BIBLIOGRAPHY OF RAILWAYS[1]

PUBLISHED UNDER THE SUPERVISION OF

P. GHILAIN,

General Secretary of the Permanent Commission of the International Railway Congress Association.

(JANUARY 1952)

016. 385. (02]

I. — BOOKS.

In French.

1951 621 .86 & 621 .87

Les appareils de levage et de manutention. Commande. Description et Utilisation. Numéro spécial de la *Technique Moderne* (Mars 1951).

Paris (VI[e]), Dunod, éditeur, 92, rue Bonaparte. Un volume (24.5 × 31.5 cm) de 212 pages avec de nombreuses illustrations. (Prix : 960 fr. fr.)

1951 385 (08 (493)

COMPAGNIE BELGE DE CHEMINS DE FER ET D'ENTREPRISES.

Rapport à l'assemblée générale ordinaire du 11 avril 1951. **Exercice 1950.** 70[e] exercice social.

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Verkehr, Nr 32, 11. August, S. 977.
Der **Stand des Koordinationsproblems.** (2 000 Wörter.)

1951 **656** .235 .5
Verkehr, Nr 37, 15. September, S. 1149.
« **Bahnsammelgut** » **in den europäischen Gütertarifen.** (2 000 Wörter.)

In English.

The Engineer. (London.)

1951 **621** .33
The Engineer, March 30, p. 412; April 6, p. 445.
Survey of existing **railway electrification systems.** (3 000 words & tables.)

1951 **621** .132 .1 (42)
The Engineer, April 20, p. 501; April 27, p. 535.
NOCK (O. S.). — Present-day **locomotive working** in Great Britain. (5 800 words, tables & fig.)

1951 **625** .233 (42)
The Engineer, May 4, p. 574.
Cold cathode lighting in a railway coach. (500 words & fig.)

1951 **621** .132 .1 (42)
The Engineer, May 4, p. 576.
Standard 4-6-0 mixed traffic locomotive. (2 000 words & fig.)

1951 **621** .132 .1 (42)
The Engineer, May 18, p. 653.
POULTNEY (E. C.). — **Characteristics of the first standard locomotive** for British Railways. (3 000 words & fig.)

1951 **621** .131 .3 (42)
The Engineer, May 25, p. 691.
NOCK (O. S.). — **The « Britannia » class locomotives on test.** (2 400 words & fig.)

1951 **621** .431 .72 (42)
The Engineer, June 1, p. 725.
Diesel-mechanical locomotive No. 10 100 (4 200 words & fig.)

1951 **656** .25 (42)
The Engineer, June 8, p. 749; June 15, p. 783.
Resignalling at York. (8 400 words & fig.)

1951 **621** .431 .72 (42)
The Engineer, June 8, p. 765.
A rail traction Diesel-electric unit. (1 800 words & fig.)

Engineering. (London.)

1951 **625** .144 .4 (42)
Engineering, March 30, p. 367.
Optical versine gauge for railway curves. (400 words & fig.)

1951 **625** .232 (42)
Engineering, March 30, p. 371.
Standard coaches for British Railways. (1 000 words & fig.)

1951 **621** .33 (42)
Engineering, March 30, p. 387; April 13, p. 443.
Electrification of railways in Great Britain. (4 000 words, tables & map.)

1951 **621** .132 (42)
Engineering, May 4, p. 528.
Standard 4-6-0 mixed traffic locomotive, British Railways. (2 100 words & fig.)

1951 **621** .131 .3 (42)
Engineering, May 25, p. 631.
Mobile and stationary tests of locomotives. (2 200 words & fig.)

1951 **621** .133 (42)
Engineering, June 1, p. 645.
POULTNEY (E. C.). — **Boiler performance** of a Mallet locomotive. (5 600 words, tables & fig.)

1951 **621** .431 .72 (42)
Engineering, June 1, p. 657.
Diesel locomotive with « Fell » mechanical drive. (1 100 words & fig.)

1951 **656** .25 (42)
Engineering, June 8, p. 688.
Re-signalling at York railway station. (2 200 words & fig.)

1951 **625** .23 (42)
Engineering, June 8, p. 705.
Light alloy coach for London Transport surface lines. (3 000 words & fig.)

The Journal of the Institute of Transport. (London.)

1951 **656** .23
The Journal of the Institute of Transport, July, p. 179.
WOOD (W. V.). — **Transport efficiency and levels of charges.** (5 000 words.)

1951 **385** .6
The Journal of the Institute of Transport, July, p. 193.
COTTIER (R.). — **Sixty years of the Berne International Railway Conventions.** (3 500 words.)

1951 **625** .24
Journal of the Institute of Transport, September, p. 211.
PARKHOUSE (S. E.). — **Railway freight rolling stock.** (8 000 words.)

1951 **656** (42)
Journal of the Institute of Transport, September, p. 219.
TAYLOR (N. S.). — Some London **freight problems.** (5 000 words.)

Journal, The Permanent Way Institution. (London.)

1951 **625** .13 (492)
Journal, The Permanent Way Institution, August, p. 130.
CUPERUS (J. L. A.). — **Civil engineering works on the Netherlands Railways after the war.** (5 000 words & fig.)

The Locomotive. (London.)

1951 **625** .23 (42)
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B. R. standard coaches. (700 words & fig.)

1951 **621** .134 .3 (42)
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British-Caprotti valve gear. (800 words & fig.)

1951 **621** .132 .1 (42)
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British Railways standard locomotives. (1 200 words & fig.)

1951 **621** .132 .1 (43)
The Locomotive, June 15, p. 95.
German 2-6-2 passenger locomotive. (300 words & fig.)

Modern Transport. (London.)

1951 **656** .211 (42)
Modern Transport, July 7, p. 5.
Train arrival bureau at Euston. (600 words & fig.)

1951 **625** .173 (42)
Modern Transport, July 21, p. 11.
Track-laying cranes. (600 words & fig.)

1951 **621** .132 .1 (42) & **621** .132 .6 (42)
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1951 **625** .232 (42)
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1951 **621** .431 .72 (41)
Modern Transport, August 4, p. 3.
Diesel conversion of railway coaches. (1 500 words & fig.)

Railway Age. (New York.)

1951 **656** .212 .5 (73)
Railway Age, June 11, p. 52.
Retarders are used in this small yard. (2 500 words & fig.)

1951 **625** .232 (73)
Railway Age, June 11, p. 57.
New Great Northern passenger cars. (3 600 words & fig.)

1951 **621** .438 (73)
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Railway Engineering and Maintenance. (Chicago.)

1951 **625** .143 .2
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1951 **625** .17 (73)
Railway Engineering and Maintenance, August, p. 703.
Detail programming of track work gets results on D. L. & W. (4 000 words & fig.)

Railway Gazette. (London.)

1951 **625** .24 (73)
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Laminated plywood in van building. (2 400 words & fig.)

1951 **656** .284 (42)
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1951 **621** .132 .1 (42)
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British Railways class « 4 » standard locomotives. (800 words & fig.)

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Progress of new woodhead tunnel. (1 500 words & fig.)

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Buffer shunting tractor for railway wagons. (400 words & fig.)

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1951 **625** .173 (42)
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1951 **625** .245 (44)
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Shock absorber for goods wagons. (300 words & fig.)

1951 **656** .212 (73)
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1951 **621** .338 (42)
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1951 **621** .431 .72 (593)
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American three-car train for Colombia. (1 200 words & fig.)

In Danish (= 439.81).

Ingeniøren. (Copenhagen.)

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ANDERSEN (J.). — **Electrification of the lines of the Danish State Railways.** (4 300 words.)
BENDTSEN (R.). — Reply. (1 500 words.)

In Spanish.

Ferrocarriles y Tranvias. (Madrid.)

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DE LASALA MILLARUELO (J.). — La **situación de los ferrocarriles europeos** comentada en relación con los ferrocarriles españoles. (8 000 palabras, cuadros & fig.)

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BOLOMEY (J.). — Contribución al estudio del **hormigón precomprimido.** (6 000 palabras & fig.)

In Italian.

Ingegneria ferroviaria. (Roma.)

1951 656 .256

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RIGHI (R.). — I **circuiti di binario** e loro condizioni di impianto e di esercizio in territorio elettrificato a corrente continua. (7 000 parole & fig.)

1951 621 .33 (45) 656 .254 (45) & 656 .257 (45)

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1951 621 .132 .8 (73) & 621 .438 (73)

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Trasporti pubblici. (Roma.)

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1951 385 .15 (45)

Trasporti pubblici, aprile, p. 405.

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1951 621 .431 .72 (45)

Trasporti pubblici, aprile, p. 447.

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1951 625 .2 : 625 .62 (45)

Trasporti pubblici, aprile, p. 451.

Vetture tranviarie articolate per la città di Torino. (4 000 parole & fig.)

In Polish (= 91.885).

Przeglad Kolejowy (Warsaw.)

1949 656 .212 .5 = 91.885

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In Portuguese.

Gazeta dos Caminhos de ferro. (Lisboa.)

1951 621 .392 & 625 .144 .1

Gazeta dos Caminhos de ferro, nº 1520, 16 de abril, p. 63.

MACHADO DE CARVALHO (V.). — Algumas considerações sobre a **soldadura de carris.** (3 500 palavras.)

1951 **656** (469)

Gazeta dos Caminhos de ferro, nº 1522, 16 de Maio, p. 103; nº 1523, 1 de Junho, p. 119; nº 1525, 1 de Julho, p. 169.

Coordenacâo dos Transportes terrestres. O Governo concedeu à C. P. a exploraçâo, por 50 anos, das linhas férreas nacionais e um empréstimo de 320.000 contos, pelo Fundo de fomento nacional. (30 000 palavras.)

1951 **385**

Gazeta dos Caminhos de ferro, nº 1529, 1 de setembro, p. 271.

BOYER (Ch.). — **O problema dos Caminhos de ferro.** (10 000 palabras.)

In Swedish = 439.71).

Järnvägs-Teknik. (Stockholm.)

1951 **656** .254 (485) = 439.71

Järnvägs-Teknik, No. 3, p. 70.

LINDEROTH (M.). — **A new Swedish system of automatic telephone.** (3 400 words & fig.)

Meddelandan från Svenska Lokaltrafik Föreningen. (Stockholm.)

1951 **625** .42 = 439.71

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ZETTERSTRÖM (O.). — Are the towns of Lisbon, Cleveland and Zurich likely to have schemes for **underground railways,** as towns of millions of inhabitants. (2 200 words & fig.)

1951 **621** .335 (438) = 439.71

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ASPENBERG (E.). — **Fast Swedish trains used for the suburban traffic in Warsaw.** (2 000 words & fig.)

Teknisk Tidskrift. (Stockholm.)

1951 **621** .33 = 439.71

Teknik Tidskrift, No. 4, p. 277.

ULLERFORS (K.). — **Dynamometer car for the measure of contact lines.** (1 500 words & fig.)

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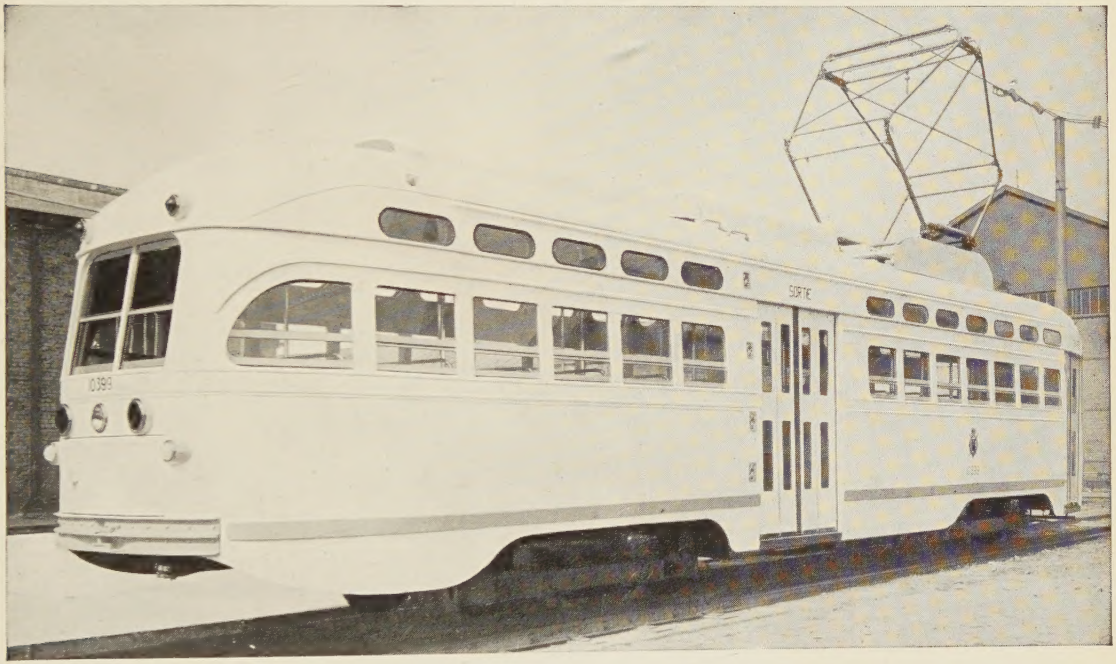

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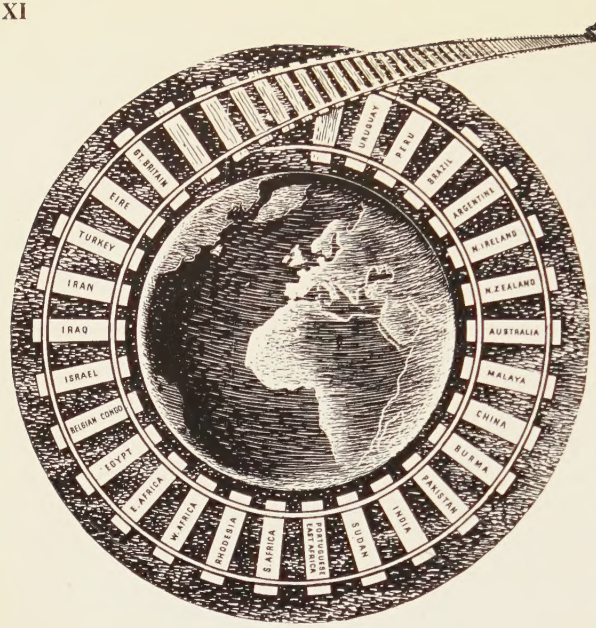

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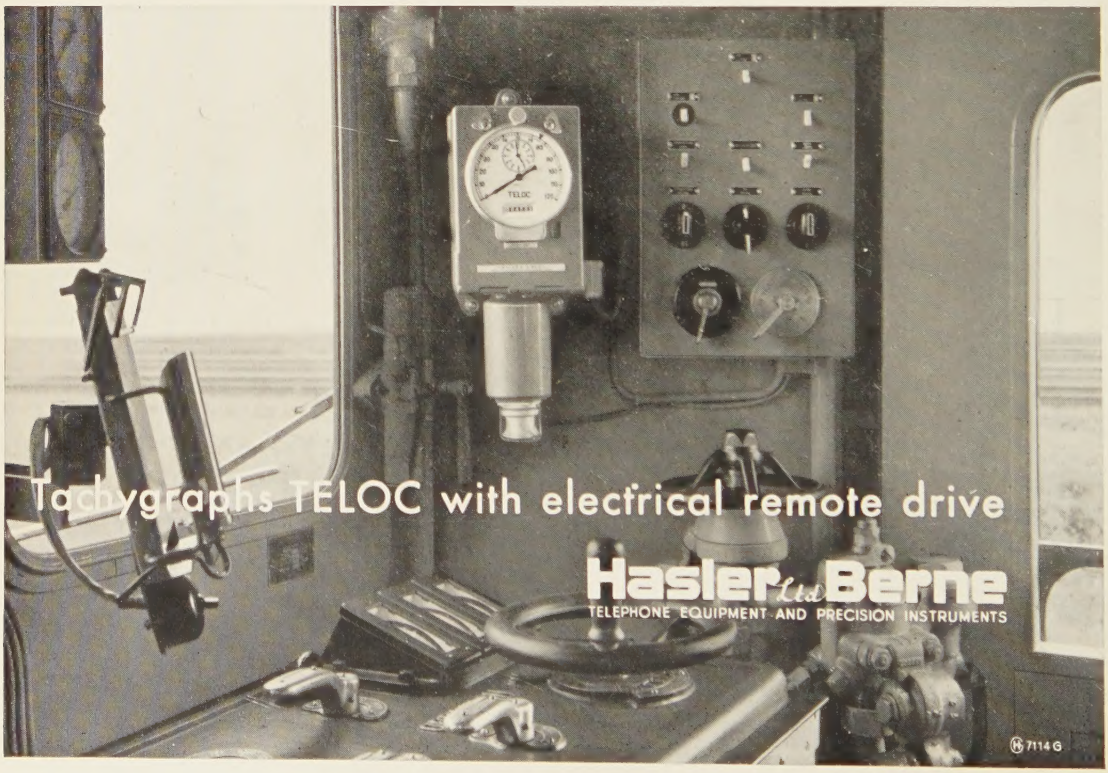

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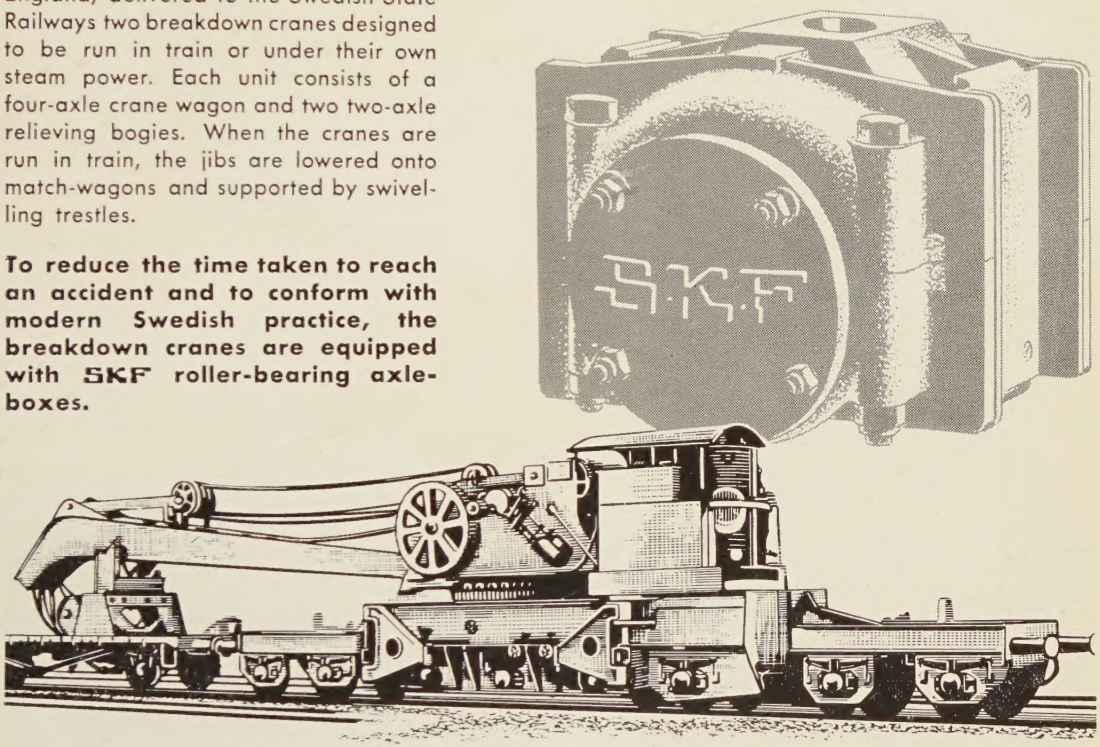

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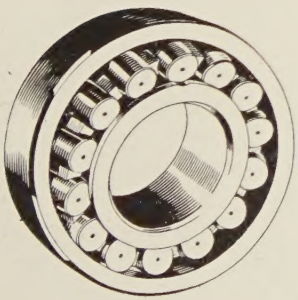

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